On new completely regular q-ary codes[1]

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Abstract

In this paper from q-ary perfect codes new completely regular q-ary codes are constructed. In particular, two new ternary completely regular codes are obtained from ternary Golay $[11, 6, 5]$ code. The first $[11, 5, 6]$ code with covering radius $\rho = 4$ coincides with the dual Golay code and its intersection array is $(22, 20, 18, 2, 1; 1, 2, 9, 20, 22)$. The second $[10, 5, 5]$ code, with covering radius $\rho = 4$, coincides with the dual code of the punctured dual Golay code and has the intersection array given by $(20, 18, 4, 1; 1, 2, 18, 20)$.

New q-ary completely regular codes are obtained from q-ary perfect codes with $d = 3$. It is shown that under certain conditions a q-ary perfect $(n, N, 3)$ code gives a new q-ary completely regular code with $d = 4$, covering radius $\rho = 3$ and intersection array $(n(q-1), (n-1)(q-1), 1; 1, (n-1), n(q-1))$. For the case $q = 2^m$, $(m \geq 2)$ this gives, in particular, an infinite family of new q-ary completely regular $[q + 1, q - 2, 4]$ codes with covering radius $\rho = 3$ and with intersection array $(q^2 - 1, q(q - 1), 1; 1, q, q^2 - 1)$. Any q-ary perfect $(n, N, 3)$ code gives a new completely regular $(n-1, N/q, 3)$ code with covering radius $\rho = 2$ and intersection array $((n - 1)(q - 1), (q - 1); 1, (n - 1(q - 1))$.

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1 Introduction

Let F be arbitrary finite alphabet of size q with elements, denoted by $\{0, 1, ..., q-1\}$, assuming that F is at least an abelian group with an addition operation denoted by "+" (and inverse operation "-") and with zero element denoted by 0. Let $\mathrm{wt}(\mathbf{v})$ denote the *Hamming weight* of a vector $\mathbf{v} \in F^n$ (i.e. the number of its nonzero positions), and $d(\mathbf{v}, \mathbf{u}) = \mathrm{wt}(\mathbf{v} - \mathbf{u})$ denote the *Hamming distance* between two vectors $\mathbf{v}$ and $\mathbf{u}$. By the same way (i.e. "+"/"-") we denote the component-wise addition/substruction of vectors of F^n). A q-ary $(n, N, d)_q$-code C is a subset of F^n where n is the *length*, d is the *minimum distance*, and $N = |C|$ is the *cardinality* of C. For the case when F is a finite field $\mathbb{F}_q$ and C is a k-dimensional linear subspace of $\mathbb{F}_q^n$, C is a *linear* code, denoted $[n, k, d]_q$, where $N = q^k$.

Given any vector $\mathbf{v} \in F^n$, its *distance to the code* C is

$$d(\mathbf{v}, C) = \min_{\mathbf{x} \in C} \{d(\mathbf{v}, \mathbf{x})\},$$

and the *covering radius* of the code C is

$$\rho = \max_{\mathbf{v} \in F^n} .\{d(\mathbf{v}, C)\}$$

For a given q-ary code C with covering radius $\rho = \rho(C)$ define

$$C(i) \ = \ \{\mathbf{x} \in F^n : \ d(\mathbf{x}, C) = i\}, \ \ i = 1, 2, ..., \rho.$$

We assume that a q-ary code C always contains the zero vector, if it is not stated the opposite. Let $D = C + \mathbf{x}$ be a *shift* of C. The *weight* $\mathrm{wt}(D)$ of D is the minimum weight of the codewords of D. For an arbitrary shift D of weight $i = \mathrm{wt}(D)$ denote by $\mu(D) = (\mu_0(D), \mu_1(D), ..., \mu_n(D))$ its weight distribution, where $\mu_i(D)$ denotes the number of words of D of weight i. Denote by C_j (respectively, D_j, and $C(i)_j$) the subset of C (respectively, of D and $C(i)$), formed by all words of the weight j. In our terminology $\mu_i(D) = |D_i|$.

Definition 1 *A q-ary code C with covering radius ρ is called* completely regular *if the weight distribution of any shift D of weight i, $i = 0, 1, ..., \rho$ of C is uniquely defined by the minimum weight of D, i.e. by the number $i = wt(D)$.*

Let C be a q-ary e-error-correcting code, i.e. a code with $d \geq 2e + 1$. It has been conjectured for a long time that if C is a completely regular code and $|C| > 2$, then $e \leq 3$. Moreover, in [10] it is conjectured that the only completely regular code C with $|C| > 2$ and $d \geq 8$ is the well known extended binary Golay $(24, 2^{12}, 8)$ code with $\rho = 4$. As we know from the results [14, 17] for the case $\rho = e$ and [13, 15, 8]) for the case $\rho = e + 1$, any such nontrivial unknown code should have a covering radius $\rho \geq e + 2$.

This paper is a natural continuation of our previous papers [4] and [5], where we derived many new completely regular and completely transitive binary codes, and described all non-antipodal binary such codes. Moreover in [4], we have disproved the conjecture of Neumaier above (see [10]), finding new completely regular binary $[23, 11, 8]$-code with covering radius $\rho = 7$ and with intersection array $(23, 22, 21, 20, 3, 2, 1; 1, 2, 3, 20, 21, 22, 23)$.

Our purpose in this paper is to prove the existence of new q-ary completely regular codes. In particular, we proved that the ternary $[11, 5, 6]$-code, which is a third part of the perfect ternary Golay $[11, 6, 5]$-code is a completely regular code with minimal distance 6, with covering radius $\rho = 5$ and with intersection array $(22, 20, 18, 2, 1; 1, 2, 9, 20, 22)$. Puncturing of this code also gives a new completely regular code with minimal distance 5, with covering radius $\rho = 4$ and with intersection array $(20, 18, 4, 1; 1, 2, 18, 20)$. In fact, in Section 3, we prove that the shortened Golay code is completely regular, but this shortened code is equivalent to that punctured.

A q-th part of a q-ary perfect $(n = (q^m - 1)/(q - 1), N = q^{n-m}, 3)$-code with Hamming parameters gives under certain conditions a new completely regular code with $d = 4$ and $\rho = 3$ and intersection array $((q - 1)n, (q - 1)(n - 1), 1; 1, n - 1, (q - 1)n)$. Furthermore, any q-ary perfect $(n, N, 3)$-code gives a new q-ary completely regular $(n - 1, N/q, 3)$ code with $\rho = 2$ and intersection array $((q - 1)(n - 1), q - 1; 1, (q - 1)(n - 1))$.

All these new codes are uniformly packed in the wide sense, i.e. in the sense of [1, 2].

The paper is organized as follows. In Section 2 we give some preliminary results concerning q-ary completely regular codes. In Section 3 we give new q-ary completely regular codes, obtained from ternary perfect Golay code. Section 4 is dedicated to new q-ary completely regular codes, obtained from q-ary 1-perfect codes.

2 Preliminary results

We give some definitions, notations and results which we will need. Given two sets $X, Y \subset F^n$, define their minimum distance $d(X, Y)$:

$$d(X, Y) \;=\; \min\{d(\mathbf{x}, \mathbf{y}) : \; \mathbf{x} \in X, \; \mathbf{y} \in Y\}.$$

We write $X + \mathbf{x}$ instead of $X + \{\mathbf{x}\}$. For a given vector $\mathbf{x} \in F^n$ let $\bar{\mathbf{x}}$ denote any of q vectors at distance n from $\mathbf{x}$, i.e. $d(\mathbf{x}, \bar{\mathbf{x}}) = n$).

Definition 2 *Let C be a q-ary code of length n and let ρ be its covering radius. We say that C is* uniformly packed *in the wide sense, i.e. in the sense of [1], if there exist rational numbers $\alpha_0, \ldots, \alpha_\rho$ such that for any $\mathbf{v} \in F^n$*

$$\sum_{k=0}^{\rho} \alpha_k \, f_k(\mathbf{v}) \;=\; 1 \;, \tag{1}$$

where $f_k(\mathbf{v})$ is the number of codewords at distance k from $\mathbf{v}$.

We use also the definition of completely regularity given in [10].

Definition 3 *A code C is completely regular if, for all $l \geq 0$, every vector $x \in C(l)$ has the same number c_l of neighbors in $C(l-1)$ and the same number b_l of neighbors in $C(l+1)$. Also, define $a_l = (q-1)n - b_l - c_l$ and note that $c_0 = b_\rho = 0$. Define by $\{b_0, \ldots, b_{\rho-1}; c_1, \ldots, c_\rho\}$ the intersection array of C and by L the intersection matrix of C:*

$$L \;=\; \begin{pmatrix} a_0 & b_0 & 0 & 0 & \cdots & 0 \\ c_1 & a_1 & b_1 & \cdots & 0 & 0 \\ 0 & c_2 & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \vdots & \ddots & \ddots & b_{\rho-1} \\ 0 & 0 & \cdots & \cdots & c_\rho & a_\rho \end{pmatrix}.$$

The *support* of $\mathbf{v} \in F^n$, $\mathbf{v} = (v_1, \ldots, v_n)$ is $supp(\mathbf{v}) = \{ \ \ell \mid v_\ell \neq 0 \ \}$. Say that a vector $\mathbf{v}$ *covers* a vector $\mathbf{z}$ if the condition $z_i \neq 0$) implies $z_i = v_i$.

Following [16], we use the following *q-ary t-designs*.

Definition 4 *A set T of vectors $\mathbf{v} \in F^n$ of weight w is a q-ary t-*design, *denoted $T(n,w,t,\lambda)_q$, if for any vector $\mathbf{z} \in F^n$ of weight t, $1 \leq t \leq w$, there are precisely λ vectors $\mathbf{v}_i$, $i = 1, ..., \lambda$, from $T(n,w,t,\lambda)_q$, each of them covering $\mathbf{z}$. If $\lambda = 1$ we call this t-design by a q-ary Steiner system and denote by $S(n,w,t)_q$.*

Definition 5 *Let F be an abelian group under addition. We say that a vector $\mathbf{x} = (x_1, ..., x_n) \in F^n$ has parity s, $s \in F$, if*

$$\sum_{i=1}^{n} x_i = s.$$

Definition 6 *We say that a q-ary code C with minimum distance d and with zero codeword has t-design property, if any nonempty set C_j, $d \leq j \leq n$, is a q-ary t-design.*

The following well known fact directly follows from the definition of completely regular code.

Lemma 1 *Let C be a completely regular code with minimum distance d and with zero codeword. Then C has t-design property, where $t = e$, if $d = 2e+1$ and $t = e+1$, if $d = 2e+2$.*

For perfect codes with $d = 2e + 1$ this statement can be formulated as follows.

Lemma 2 *Let C be a q-ary perfect code with minimum distance $d = 2e+1$ and with zero codeword. Then C has t-design property, where $t = e+1$. Furthermore, the set C_d is a q-ary Steiner system $S(n,d,e+1)_q$.*

Proof. For C_d the result is straightforward. Indeed, by definition of perfect code any vector $\mathbf{x} \in F^n$ of weight $e+1$ should be covered by exactly one codeword of weight d, which implies that C_d is a Steiner system $S(n, d, e+1)_q$. For larger weights it can be done using the same arguments as in [13] for the binary case. $\triangle$

The next statement can be found in [10] for binary codes. For the case $q > 2$ it can be proved easily.

Lemma 3 *If C is completely regular with covering radius ρ, then $C(\rho)$ is also completely regular, with reversed intersection array and distribution diagram.*

Next three Lemmas will be needed in the following Section.

Lemma 4 *Let C be a q-ary linear code and denote by $C^{(s)}$ the shortened code of C formed by taking the codewords of C which have a fixed coordinate equal to zero and then by deleting this fixed coordinate. Let $C^{\perp}$ be the dual code of C and $C^{(p)}$ the punctured code of C, i.e. obtained from C by deleting the fixed coordinate. Then*

$$C^{(s)\perp} = C^{\perp(p)}$$

Proof. It is straightforward. $\triangle$

Lemma 5 *Let C be the Golay code (binary or ternary). Then, $C^{(0)}$, the subcode of C formed by all codewords with parity zero coincides with the dual code of C.*

Proof. The Golay code (binary or ternary) is a cyclic, quadratic-residue code C of length, respectively, $n = 23$ and $n = 11$. We know (see [9]) that

$x^n - 1 = (x-1)\cdot g(x)\cdot h(x)$, where $g(x)$ is the generator polynomial of C and $h(x)$ is the reciprocal polynomial of $g(x)$.

Let $C^{(0)}$ be the subcode of C formed by all codewords with parity zero. Code $C^{(0)}$ is a cyclic code with generator matrix $(x-1)\cdot g(x)$ and the dual code $C^{(0)\perp}$ is a cyclic code with generator polynomial given by the reciprocal polynomial of $h(x)$ which coincides with $g(x)$, so $C^{(0)} = C^{\perp}$. $\triangle$

Lemma 6 *Let C be the Golay code (binary or ternary). Then, the two codes $C^{(0)(p)} = C^{(s)\perp}$ and $C^{(s)}$ are equivalent, but $C^{(s)}$ is not a self-dual code.*

Proof. We begin with the ternary case, so let C be the ternary Golay code. Note that $C^{(s)}$ comes from the extended Golay code $C^{(e)} = [12, 3^6, 6]$ taking the codewords with $(0,0)$, $(1,0)$ or $(2,0)$ as the first two coordinates and deleting these two coordinates. Moreover note that $C^{(0)(p)}$ also comes from the extended Golay code $C^{(e)}$ taking the codewords with $(0,0)$, $(0,1)$ or $(0,2)$ as the first two coordinates and deleting these two coordinates.

It is very well-known that code $C^{(e)}$ is unique (see [9]), so the two constructions above are equivalent. Moreover, looking at the two generator matrices of codes $C^{(s)\perp}$ and $C^{(s)}$ it is easy to see that $C^{(s)}$ is not a self-dual code.

Concerning the binary Golay case the proof is the same and we do not repeat it.

3 New q-ary completely regular codes from ternary Golay code

As we mentioned already the even, or odd half subcodes of a binary perfect code, and codes, obtained by puncturing of these subcodes give new completely regular codes. For q-ary perfect codes it is not the case, since we can not guarantee the existence of subcodes with minimum distance $d = 4$ for perfect codes with Hamming parameters.

We start from the ternary Golay code. Before we consider some q-ary designs, arising from the ternary Golay code.

Lemma 7 *Let G be the ternary perfect Golay $[11, 6, 5]_3$-code. Denote by $G^{(0)}$ the subcode of G with minimum distance* 6*, formed by all codewords with zero overall parity checking modulo* 3*. Then:*
(i) *Code $G^{(0)}$ is formed by the zero vector and all the codewords of G with weights* 6 *and* 9*.*
(ii) *The set G_5 is a ternary Steiner system $S(11, 5, 3)_3$ and the sets $G_6^{(0)}$ and $G_9^{(0)}$ are ternary* 3*-designs $T(11, 6, 3, 2)_3$ and $T(11, 9, 3, 3)_3$ respectively.*

Proof. (i) It can be checked directly from construction of the ternary Golay code.

(ii) By Lemma 2 the set G_5 is a ternary Steiner system $S(11, 5, 3)$. Now by the same lemma the sets G_6 and G_9 are ternary 3-design. Taking into account, that all codewords from G_6 have the zero parity, we conclude that $G_6^{(0)} = G_6$. Since $|G_6^{(0)}| = 132$, taking into account that it is a ternary 3-design, we obtain that $G_6^{(0)}$ is $T(11, 6, 3, 2)_3$. Similarly, since $G_9^{(0)} = G_9$ (all codewords from G_9 have the zero parity) and since $|G_9^{(0)}| = 110$, we deduce that $G_9^{(0)}$ is $T(11, 9, 3, 3)_3$. $\triangle$

Theorem 1 *Let G be the ternary perfect Golay $[11, 6, 5]_3$-code. Denote by $G^{(0)}$ the subcode of G with minimum distance 6, formed by all codewords with zero overall parity checking modulo 3. Then:*
(i) $G^{(0)}$ is the $[11, 5, 6]$ code, dual to G.
(ii) $G^{(0)}$ is the completely regular code with covering radius 5 and with intersection array $\{22, 20, 18, 2, 1; 1, 2, 9, 20, 22\}$.
(iii) $G^{(0)}$ is uniformly packed in the sense of [1] with parameters α_i:

$$\alpha_0 = 1,\ \alpha_1 = \frac{5}{11},\ \alpha_2 = \frac{29}{110},\ \alpha_3 = \frac{41}{330},\ \alpha_4 = \frac{17}{330},\ \alpha_5 = \frac{1}{66}.$$

Proof. (i) By Lemma 5 the code $G^{(0)}$ coincides with the dual of G. From the other side, the extension of G is a self-dual code (Theorem 16.19 in [9]). Hence $G^{(0)}$ is $[11, 5, 6]$ formed by all codewords with parity zero.

(ii) Let $\mathbf{x} \in \mathbb{F}_3^{11}$ and $\mathbf{c} \in G^{(0)}$. Since $G^{(0)}$ is a ternary code with minimum distance $d = 6$ and covering radius 5, for the case $d(\mathbf{x}, \mathbf{c}) = i$, where $i = 0, 1, 2$ we have clearly that

$$a_i = (q-2)i = i,\ b_i = 22 - (q-1)i = 22 - 2i,\ c_i = i.$$

Consider the case $i = 3$. Let $\mathbf{x}$ be a vector of weight 3 and let $W(\mathbf{x})$ be a sphere of radius one with center $\mathbf{x}$. Since $G_6^{(0)}$ is a ternary 3-design $T(11, 6, 3, 2)_3$, for any vector $\mathbf{x} \in \mathbb{F}_3^{11}$ of weight three there exist exactly two codewords, say $\mathbf{z}_1$ and $\mathbf{z}_2$ from $G_6^{(0)}$ of weight 6 and one codeword, say $\mathbf{v}$ from G_5, which cover $\mathbf{x}$. Since $d(\mathbf{z}_1, \mathbf{z}_2) = 6$ and $d(\mathbf{z}_i, \mathbf{v}) = 5$ for $i = 1, 2$, all these three vectors $\mathbf{z}_1$, $\mathbf{z}_2$ and $\mathbf{v}$ have intersection only on $\mathrm{supp}(\mathbf{x})$. Hence, the coordinate set $J = \{1, 2, ..., 11\}$ of G is partitioned into four disjoint subsets, namely,

$$J = \mathrm{supp}(\mathbf{x}) \cup \mathrm{supp}(\mathbf{v} - \mathbf{x}) \cup \mathrm{supp}(\mathbf{z}_1 - \mathbf{x}) \cup \mathrm{supp}(\mathbf{z}_2 - \mathbf{x}). \qquad (2)$$

Now let $\mathbf{y} \in W(\mathbf{x})$. We consider four cases $j)$, $j = 1, 2, 3, 4$, denoting by $a_{3,j}, b_{3,j}, c_{3,j}$ the contributions to intersection numbers a_3, b_3, c_3 for the case $j)$.

1). $\mathrm{wt}(\mathbf{y}) = 2$, $\mathrm{supp}(y) \subseteq \mathrm{supp}(\mathbf{x})$. For this case, we obtain that $c_{3,1} = 3$.

2). $\mathrm{wt}(\mathbf{y}) = 3$, $\mathrm{supp}(y) \subseteq \mathrm{supp}(\mathbf{x})$. We deduce here that $a_{3,2} = 3(q-2) = 3$.

3). $\mathrm{wt}(\mathbf{y}) = 4$, $\mathrm{supp}(y) \subset \mathrm{supp}(\mathbf{v})$. First, assume that $\mathbf{y}$ is covered by $\mathbf{v}$. Since $\mathbf{v}$ has the weight 5, it occurs for two distinct vectors $\mathbf{y}$. Hence we obtain, that $b_{3,3} = 2$. Assume now that $\mathbf{y}$ is not covered by $\mathbf{v}$. We claim that $\mathbf{y}$ belongs to $G^{(0)}(3)$. Indeed, let $\mathbf{x}_1$ of weight three obtained from $\mathbf{y}$ by deleting the first nonzero positions of $\mathbf{x}$. Now $\mathbf{x}_1$ should be covered by one codeword, say $\mathbf{v}_1$ from G_5. But $d(\mathbf{v}, \mathbf{v}_1) \geq 5$, which implies that $\mathbf{v}_1$ should have the zero element on the first position of $\mathbf{x}$ (if not $\mathbf{v}$ and $\mathbf{v}_1$ will be at distance 4 from each other). Now, there are two codewords, say $\mathbf{z}_3$ and $\mathbf{z}_4$, covering $\mathbf{x}_1$. Recalling the partition (2) (i.e. four sets $\mathrm{supp}(\mathbf{x}_1)$, $\mathrm{supp}(\mathbf{v}_1 - \mathbf{x}_1)$, $\mathrm{supp}(\mathbf{z}_3 - \mathbf{x}_1)$, and $\mathrm{supp}(\mathbf{z}_4 - \mathbf{x}_1)$ partition the set J), we conclude, therefore, that one of these words $\mathbf{z}_3$ or $\mathbf{z}_4$ will have a nonzero element on the first position of $\mathbf{x}$. We conclude that $a_{3,3} = 2(q-2) = 2$.

4). $\mathrm{wt}(\mathbf{y}) = 4$, $\mathrm{supp}(y) \subset \mathrm{supp}(\mathbf{z}_i)$, $i \in \{1, 2\}$. If $\mathbf{y}$ is covered by some z_s, $s \in \{1, 2\}$, this means that $\mathbf{y} \in G^{(0)}(2)$, which implies that $c_{3,4} = 2 \cdot |\mathrm{supp}(\mathbf{z}_s)| = 2 \cdot 3 = 6$. If $\mathbf{y}$ is not covered by $\mathbf{z}_s$, this means that $\mathbf{y} \in G^{(0)}(3)$, which gives $a_{3,4} = 2 \cdot |\mathrm{supp}(\mathbf{z}_s)|(q-2) = 2 \cdot 3(q-2) = 6$.

Thus all four cases give:

$$c_3 = 6 + 3 = 9,\ a_3 = 3 + 2 + 5 = 11,\ b_3 = 2.$$

For the case $i = 4, 5$ we deduce using Lemma 3

$$a_i = b_i = 5 - i,\ \ c_i = 22 - 2(5 - i).$$

(iii) The parameters α_i come from the equation (1) and Lemma 7. Since for the code G we have that $\rho = 5$ and $d = 6$ we obtain $\alpha_0 = 1$.

Now we find α_5. Take as $\mathbf{x}$ any vector from G_5. We see that

$$f_5(\mathbf{x}) \ = \ \frac{|G_5|}{q-1} \ = \ \frac{1}{2}\,|G_5| \ = \ 66.$$

This gives

$$\alpha_5 = \frac{1}{f_5(\mathbf{x})} = \frac{1}{66}.$$

For α_1 we have, taking $\mathbf{x}$ of weight one:

$$\alpha_1 \cdot f_1(\mathbf{x}) + \alpha_5 \cdot f_5(\mathbf{x}) = 1. \tag{3}$$

Clearly $f_1(\mathbf{x}) = 1$. Since $G_6^{(0)}$ is a 3-design $T(11, 6, 3, 2)$, there are

$$\frac{6}{(q-1)n}\,|T(11, 6, 3, 2)| \ = \ 36$$

codewords from $G_6^{(0)}$ covering $\mathbf{x}$. Using α_5 in (3), we deduce that $\alpha_1 = 5/11$.

For $\mathbf{x}$ of weights $2, 3, 4$ we have respectively

$$\alpha_2 \cdot 1 + \alpha_4 \cdot 9 + \alpha_5 \cdot 18 = 1, \tag{4}$$

$$\alpha_3 \cdot 3 + \alpha_4 \cdot 6 + \alpha_5 \cdot 21 = 1, \tag{5}$$

and

$$\alpha_4 \cdot 15 + \alpha_5 \cdot 15 = 1, \tag{6}$$

From (6) we obtain that $\alpha_4 = 17/330$, and using this in (4) and in (5) we obtain α_2 and α_3 given in the statement. $\triangle$

Lemma 8 *Let G be the ternary perfect Golay $[11, 6, 5]_3$-code. Let $G^{(s)}$ be the $[10, 5, 5]$ code, obtained by shortening of G and let $G^{(s)}(\rho)$ be the covering set of $G^{(s)}$. Then the sets $G^{(s)}_5$, $G^{(s)}_6$ and $G^{(s)}(\rho)_4$ are ternary 2-designs $T(10, 5, 2, 4)_3$, $T(10, 6, 2, 8)_3$ and $T(10, 4, 2, 2)_3$ respectively.*

Proof. This follows directly from Lemma 7. $\triangle$

Theorem 2 *Let G be the ternary perfect Golay $[11, 6, 5]_3$-code. Denote by $G^{(s)}$ the $[10, 5, 5]$ code, obtained by shortening of G. Then:*
(i) $G^{(s)}$ is a completely regular code with covering radius 4 and with intersection array $\{20, 18, 4, 1; 1, 2, 18, 20\}$.
(ii) $G^{(s)}$ is uniformly packed in the wide sense, i.e. in the sense of [1] with parameters α_i:

$$\alpha_0 = 1,\ \alpha_1 = \frac{2}{5},\ \alpha_2 = \frac{7}{30},\ \alpha_3 = \frac{1}{12},\ \alpha_4 = \frac{1}{30}.$$

Proof. (i) Since $G^{(s)}$ is a ternary code with minimum distance $d = 5$ and with covering radius $\rho = 4$ we deduce that

$$a_i = i(q-2) = i,\ b_i = 22 - i(q-1) = 22 - 2i,\ c_i = i,\quad i = 0, 1,$$

and, from Lemma 3,

$$a_i = b_i = 4 - i,\ \ c_i = 22 - 2(4 - i),\ \ i = 3, 4.$$

Thus we have to find only these parameters for $i = 2$. Let $\mathbf{x}$ be a vector of weight two, and let $\mathbf{y} \in W(\mathbf{x})$. We consider four cases, using the same notation $a_{2,s}, b_{2,s}, c_{2,s}$ for contribution of each case $s = 1, 2, 3, 4$.

1) $\mathrm{wt}(\mathbf{y}) = 1$. Since $\mathbf{y} \in G^{(s)}(1)$ we have for this case that $c_{2,1} = 2$.

2) $\mathrm{wt}(\mathbf{y}) = 2$. Now $\mathbf{y} \in G^{(s)}(2)$ and we obtain that $a_{2,2} = 2(q-2) = 2$.

3) $\mathrm{wt}(\mathbf{y}) = 3$ and $\mathbf{y} \in G^{(s)}(3)$. Since $G^{(s)}(\rho)_4$ is $T(10,4,2,2)_3$, this happens exactly four times (indeed, two vectors of weight four from $T(10,4,2,2)_3$, covering $\mathbf{x}$ intersect each other only on $\mathrm{supp}(\mathbf{x})$). We conclude that $b_{2,3} = 2(q-1) = 4$.

4) $\mathrm{wt}(\mathbf{y}) = 3$ and $\mathbf{y} \in G^{(s)}(2)$. Here we have to show only that these cases 3) and 4) include all possible cases. Hence it is enough to show that any $\mathbf{y}$, covering $\mathbf{x}$, can not be covered by any vector from $G^{(s)}(\rho)_4$. This is clear since $\mathbf{x}$ is covered already by two vectors from $G^{(s)}(\rho)_4$ (the case 3)). Therefore we deduce that $a_{2,4} = (2 \cdot 8 - 4) = 12$. Summing up all cases we obtain that

$$a_2 = 2 + 12 = 14, \;\; b_2 = 4, \;\; c_2 = 2.$$

This finishes the first part of the proof.

(ii) Since $G^{(s)}$ is a code with $d = 5$ and $\rho = 4$ we deduce that $\alpha_0 = 1$. Choosing $\mathbf{x}$ from $G^{(s)}(\rho)_4$, we conclude that $f_4(\mathbf{x}) = |G^{(s)}(\rho)_4|/(q-1)$. Since $|G^{(s)}(\rho)_4| = |T(10,4,2,2)_3| = 60$, we obtain that $\alpha_4 = (q-1)/|G^{(s)}(\rho)_4| = 1/30$.

Now let $\mathbf{x}$ be a vector of weight one. For α_1 we have the following equation:

$$\alpha_1 \cdot f_1(\mathbf{x}) + \alpha_4 \cdot f_4(\mathbf{x}) \;=\; 1. \qquad (7)$$

We have that $f_1(\mathbf{x}) = 1$. The number $f_4(\mathbf{x})$ is the number of codewords $G_5^{(s)}$ (of weight five) covering $\mathbf{x}$, i.e. having fixed nonzero element in fixed position. Since $G_5^{(s)}$ is a ternary 2-design $T(10,5,2,4)$, we obtain for this number:

$$|T(10,5,2,4)| \cdot \frac{5}{10 \cdot 2} \; = \; 18.$$

Hence $f_4(\mathbf{x}) = 18$ and we deduce from (7) that $\alpha_1 = 2/5$.

Now taking a vector $\mathbf{x}$ of weight two, say $\mathbf{x} = (1,1,0,...,0)$, we will have the equation:

$$\alpha_2 \cdot f_2(\mathbf{x}) + \alpha_3 \cdot f_3(\mathbf{x}) + \alpha_4 \cdot f_4(\mathbf{x}) \; = \; 1. \tag{8}$$

Clearly $f_2(\mathbf{x}) = 1$. Since $G_5^{(s)}$ is a ternary 2-design $T(10,5,2,4)_3$, we have (by the definition of 2-design) that $f_3(\mathbf{x}) = 4$. But $G_6^{(s)}$ is 2-design $T(10,6,2,5)_3$. This gives a contribution of 5 for the number $f_4(\mathbf{x})$. Now taking into account $2 \cdot 4$ codewords from $G_5^{(s)}$ starting from $(1,2,...)$ and from $(2,1,...)$ (which are at distance 4 from $\mathbf{x}$), we obtain that $f_4(\mathbf{x}) = 13$. Using this in (8), we obtain that

$$\alpha_2 + 4\alpha_3 \; = \; \frac{17}{30}. \tag{9}$$

Now let $\mathrm{wt}(\mathbf{x}) = 3$ such that $\mathbf{x} \in G^{(s)}(3)$. For this case we have

$$\alpha_3 \cdot f_3(\mathbf{x}) + \alpha_4 \cdot f_4(\mathbf{x}) \; = \; 1. \tag{10}$$

By the same way we obtain for this case, that $f_3(\mathbf{x}) = 6$ and $f_4(\mathbf{x}) = 15$. From the equation above, we obtain $\alpha_3 = 1/12$ and using this in (9), we deduce that $\alpha_2 = 7/30$. $\triangle$

Note that the completely regular code $G^{(s)}$ constructed in this Theorem 2 is the dual of the punctured dual Golay code (see Lemma 4) and although $G^{(s)}$ is not self-dual is equivalent (see Lemma 6) to the punctured code of $G^{(0)}$ (the code constructed in Theorem 1).

In fact, the result of Theorem 2 holds for any q-ary perfect code with minimum distance 5. Since, for the case when q is not a prime power, the existence of such codes is an open problem, we give the next results.

Lemma 9 *Let G be a q-ary perfect $(n+1, qN, 5)_q$ code. Denote by $G^{(s)}$ the $(n, N, 5)_q$ code, obtained by shortening of G on the element 0 in the first position. Then:*
(i) The set G_5 is a q-ary Steiner system $S(n+1, 5, 3)_q$ and the set G_6 is 3-design $T(n+1, 6, 3, \gamma_6)_q$, where

$$\gamma_6 = \frac{1}{3}\left((q-1)(n-10)+6\right).$$

(ii) The sets $G_5^{(s)}$, $G_6^{(s)}$, and $G^{(s)}(4)_4$ are q-ary 2-designs $T(n, 5, 2, \beta_5)_q$, $T(n, 6, 2, \beta_6)$, and $T(n, 4, 2, q-1)_q$, respectively, where

$$\beta_5 = \frac{1}{3}(q-1)(n-4), \quad \beta_6 = \frac{1}{12}(q-1)(n-5)\left((q-1)(n-10)+6\right).$$

Proof. By definition of perfect code the set G_5 is a Steiner system $S(n+1, 5, 3)_q$ (indeed, any vector of weight 3 should be covered by exactly one codeword of G_5). Now considering all vectors of weight 4 we deduce that the set G_6 is a 3-design $T(n+1, 6, 3, \gamma_6)_q$ where γ_6 is written above. Now considering all these words from $G_5^{(s)}$ and $G_6^{(s)}$ we obtain 2-designs, given in the statement. $\triangle$

Theorem 3 *Let G be a q-ary perfect $(n+1, qN, 5)_q$ code. Denote by $G^{(s)}$ the $(n, N, 5)$ code, obtained by shortening of G on the element 0 in the first position. Then:*

(i) $G^{(s)}$ is a completely regular code with covering radius 4 and with intersection array

$$\{(q-1)n, (q-1)(n-1), 2(q-1), 1; 1, 2, (q-1)(n-1), (q-1)n\}.$$

(ii) $G^{(s)}$ is uniformly packed in the wide sense, i.e. in the sense of [1] with parameters α_i where $\alpha_0 = 1$ and

$$\alpha_1 = \frac{4}{n},\ \alpha_2 = \frac{2(2n+3q-8)}{(q-1)n(n-1)},\ \alpha_3 = \frac{6(3q-4)}{(q-1)^2 n(n-1)},\ \alpha_4 = \frac{12}{(q-1)^2 n(n-1)}.$$

Proof. (i) Since $G^{(s)}$ is a code with minimum distance $d = 5$ and covering radius $\rho = 4$ we obtain:

$$a_i = i(q-2),\ b_i = (q-1)n - i(q-1),\ c_i = i,\ \ i = 0, 1$$

and, from Lemma 3,

$$a_i = (4-i)(q-2),\ b_i = 4-i,\ c_i = (q-1)n - (4-i)(q-1),\ \ i = 3, 4.$$

Thus we have to find only these parameters for $i = 2$. Let $\mathbf{x}$ be a vector of weight two, and let $\mathbf{y} \in W(\mathbf{x})$. Since $G_5^{(s)}$ and $G^{(s)}(4)_4$ are 2-designs (see Lemma 9), there are exactly β_5 codewords $\mathbf{u}_j$, $j = 1, ..., \beta_5$ and $q-1$ vectors $\mathbf{v}_1, ..., \mathbf{v}_{q-1}$, covering $\mathbf{x}$. We consider five cases, using the same notation $a_{2,s}, b_{2,s}, c_{2,s}$ for contribution of each case $s = 1, 2, 3, 4, 5$.

1) $\mathrm{wt}(\mathbf{y}) = 1$. Since $\mathbf{y} \in G^{(s)}(1)$ we have for this case that $c_{2,1} = 2$.

2) $\mathrm{wt}(\mathbf{y}) = 2$. Now $\mathbf{y} \in G^{(s)}(2)$ and we obtain that $a_{2,2} = 2(q-2)$.

3) $\mathrm{wt}(\mathbf{y}) = 3$ and $\mathbf{y} = \mathbf{v}_s$, $s = 1, ..., q-1$. Since $G^{(s)}(\rho)_4 = G^{(s)}(4)_4$ is $T(n, 4, 2, q-1)_3$, this happens exactly $2(q-1)$ times (indeed, $q-1$ vectors of weight four from $G^{(s)}(\rho)_4$, covering $\mathbf{x}$, intersect each other only on $\mathrm{supp}(\mathbf{x})$).

We conclude that $b_{2,3} = 2(q-1)$.

4) $\mathrm{wt}(\mathbf{y}) = 3$ and $\mathrm{supp}(\mathbf{y}) = \mathrm{supp}(\mathbf{v}_s)$, $\mathbf{y} \neq \mathbf{v}_s$, $s = 1, ..., q-1$. Here we have that $d(\mathbf{y}, \mathbf{v}_s) = 2$. Denote by $\mathbf{v}_s^*$ the codeword of G_5 which results in $\mathbf{v}_s$ when we build $G^{(s)}$ from G. Let $\mathbf{y}^* = (0 \,|\, \mathbf{y})$. Then we have that $d(\mathbf{y}^*, \mathbf{v}_s^*) = 3$. We conclude that it can not be covered by any vector from $G^{(s)}(4)_4$ (since all such vectors have a nonzero first position). But G is perfect code, hence there is some codeword from G_5 covering $\mathbf{y}^*$. So, the only possibility is that it is covered by codeword of G_5, having 0 on the first position. But such words form the set $G_5^{(s)}$. Therefore, $\mathbf{y}$ is covered by some codeword of $G_5^{(s)}$. This gives $a_{2,4} = 2(q-1)(q-2)$.

5) $\mathrm{wt}(\mathbf{y}) = 3$, $\mathrm{supp}(\mathbf{y}) \neq \mathrm{supp}(\mathbf{v}_s)$, $s = 1, ..., q-1$. We claim that any such $\mathbf{y}$ is covered by some $\mathbf{u}_j$ from $G_5^{(s)}$. Indeed, using the same arguments as we used for the case 4), we can see easily, that such $\mathbf{y}$ can not be covered by any vector from $G^{(s)}(4)_4$. But, from the other side, the corresponding vector $\mathbf{y}^* = (0 \,|\, \mathbf{y})$ should be covered by some codeword of G_5. Therefore, $\mathbf{y}^*$ should be covered by some codeword from $G_5^{(s)}$. We deduce that $a_{2,4} = (n-2-2(q-1))(q-1)$. Summing up our results, we obtain that

$$a_2 = n(q-1) - 2q, \ b_2 = 2(q-1), \ c_2 = 2.$$

This finishes the proof of (i).

(ii) Since $G^{(s)}$ has the minimum distance $d = 5$ and the covering radius $\rho = 4$, we obtain that $\alpha_0 = 1$.

Now we find α_4. Taking any $\mathbf{x} \in G^{(s)}(4)_4$ we obtain that

$$\alpha_4 \cdot f_4(\mathbf{x}) \ = \ 1.$$

Let $\mathbf{x}^* = (x_0 \,|\, \mathbf{x})$, where $x_0 \neq 0$, be the corresponding vector from G_5. It is clear that the number of codewords of G_5 does not change, if we shift G by $\mathbf{x}^*$. Thus $f_4(\mathbf{x})$ is equal to the number of codewords from G_5 with fixed element x_0 on the first position, i.e. to the number $|G^{(s)}(4)_4|/(q-1)$. Since $G^{(s)}(4)_4$ is a design $T(n,4,2,q-1)$, we obtain

$$\alpha_4 \;=\; \frac{1}{f_4(\mathbf{x})} \;=\; \frac{q-1}{|G^{(s)}(4)_4|} \;=\; \frac{12}{(q-1)^2 n(n-1)}.$$

Now let $\mathbf{x}$ have weight 1. Then we have

$$\alpha_1 \cdot f_1(\mathbf{x}) + \alpha_4 \cdot f_4(\mathbf{x}) \;=\; 1.$$

We have $f_1(\mathbf{x}) = 1$ and $f_4(\mathbf{x})$ is equal to the number of codewords of $G_5^{(s)}$ with some fixed nonzero element in the first position. Since $G_5^{(s)}$ is $T(n,5,2,\beta_5)$ with $\beta_5 \;=\; \frac{1}{3}\,(q-1)(n-4)$, we obtain

$$f_4(\mathbf{x}) = \frac{1}{12} \cdot (q-1)^2 (n-1)(n-4)\,.$$

Hence

$$\alpha_1 \;=\; 1 \;-\; \alpha_4 f_4(\mathbf{x}) \;=\; \frac{4}{n}\,.$$

For any vector $\mathbf{x}$ of weight 2 we have the equation

$$\alpha_2 \cdot f_2(\mathbf{x}) + \alpha_3 \cdot f_3(\mathbf{x}) + \alpha_4 \cdot f_4(\mathbf{x}) \;=\; 1\,. \tag{11}$$

Denote by $g_j(w)$ the number of codewords of weight w of $G^{(s)}$ which are at distance j from given $\mathbf{x}$. Since $G_5^{(s)}$ is $T(n,5,2,\beta_5)$, we deduce that $g_3(5) = \beta_5$. It is also clear that $g_4(5) = 2(q-2)\beta_5$. Indeed, there are exactly $2\cdot(q-2)\cdot\beta_5$ codewords of $G_5^{(s)}$ having two nonzero positions on $\mathrm{supp}(\mathbf{x})$, where exactly one of these positions coincides with one position of $\mathbf{x}$. Now we have to consider the codewords from $G_6^{(s)}$. Since $G_6^{(s)}$ is $T(n,6,2,\beta_6)$ we obtain that $g_4(6)=\beta_6$. Thus, we have

$$f_3(\mathbf{x}) = g_3(5) = \beta_5, \quad f_4(\mathbf{x}) = g_4(5) + g_4(6) = 2(q-2)\beta_5 + \beta_6.$$

Using expressions for β_5 and β_6 in Lemma 9 we obtain from (11) the following expression:

$$\alpha_2 + \alpha_3 \cdot \frac{1}{3}\,(q-1)(n-4) \; = 1 - \alpha_4 \cdot f_4(\mathbf{x}) \; = \; \frac{2(n-1)+3(q-1)(n-3)}{(q-1)n(n-1)}. \quad (12)$$

We have one more linear equation on α_2 and α_3, coming from sphere packing conditions for uniformly packed codes [1], namely

$$\sum_{j=0}^{4} \alpha_j (q-1)^j \binom{n}{j} \; = \; \frac{q^n}{|G^{(s)}|}\,. \quad (13)$$

Since $|G^{(s)}| = |G|/q$, taking into account that G is perfect, we obtain from (13) that

$$\sum_{j=0}^{4} \alpha_j (q-1)^j \binom{n}{j} \; = \; \frac{q^{n+1}}{|G|} \; = \; \sum_{s=0}^{2} (q-1)^s \binom{n+1}{s}. \quad (14)$$

Using now known values α_j for $j=0,1$ and $j=4$, we reduce the equality above to the following expression:

$$\frac{2(3(q-1)(n-1)+(n-3))}{(q-1)n(n-1)} \; = \; \alpha_2 \; + \; \frac{1}{3}(q-1)(n-2)\alpha_3\,. \quad (15)$$

From (12) and (15) we deduce the values of α_2 and α_3. $\triangle$

4 New q-ary completely regular codes from q-ary 1-perfect codes

Now we turn to q-ary 1-perfect codes. For the case $d=3$ Lemma 2 looks as follows.

Lemma 10 *Let H be a q-ary perfect $(n, N, 3)_q$-code with zero codeword. Let H_w be the set of all codewords with weight w. Then the set H_3 is a q-ary Steiner system $S(n, 3, 2)_q$ and the set H_w, if it is nonempty, is a q-ary 2-design $T(n, w, \lambda_w)_q$, where $w = 4, ..., n$ and where λ_w can be found from the weight distribution of H. In particular,*

$$\lambda_4 \;=\; \frac{1}{2} \cdot (n(q-1) - 5q + 7)\;.$$

Theorem 4 *Let H be a q-ary perfect $(n, N, 3)_q$-code where q be any natural number and where n is odd. Let C be any subcode of H with minimum distance $d_C = 4$ and cardinality $|C| = |H|/q$ and with following property. For any choice of zero codeword in H, the set C_4 is a 2-design $T(n, 4, 2, \beta_4)_q$, where $\beta_4 = (n-3)/2$. Then:*
(i) C is a completely regular code with covering radius $\rho = 3$ and with intersection numbers

$$((q-1)n, (q-1)(n-1), 1; 1, (n-1), (q-1)n). \tag{16}$$

(ii) C is uniformly packed in the wide sense with parameters α_i:

$$\alpha_0 = 1,\; \alpha_1 = \frac{3}{n},\; \alpha_2 = \frac{2(n+2(q-2))}{(q-1)n(n-1)},\; \alpha_3 = \frac{6}{(q-1)n(n-1)}. \tag{17}$$

Proof. (i) We start with the intersection numbers of C. For the case $i = 0, 1$ we have immediately (since C has minimum distance 4 and covering radius 3)

$$a_0 = 0,\; b_0 = (q-1)n, \;\text{ and }\; c_1 = 1,\; a_1 = q-2,\; b_1 = (q-1)(n-1).$$

The case $i = 3$ is straightforward: $c_3 = (q-1)n$ and $a_3 = 0$.

Thus, we have to consider only the case $i = 2$, for which we claim that

$$a_2 = (q-2)n,\ b_2 = 1,\ c_2 = (n-1).$$

Let $\mathbf{x}$ be any vector of weight two and let $\mathbf{y} \in W(\mathbf{x})$. Since H_3 is the Steiner system $S(n,3,2)_q$, there is the vector $\mathbf{v} \in H_3$, covering $\mathbf{x}$. Similarly, since C_4 is the $T(n,4,2,\beta_4)_q$ with $\beta_4 = (n-3)/2$, there are β_4 codewords, say $\mathbf{u}_j$, $j = 1, ..., \beta_4$ all of them covering $\mathbf{x}$. Since C_4 is a code with minimum distance four and since H_3 is at distance three from C_4, we obtain the following disjoint partition of the coordinate set J of H:

$$J \ = \ \mathrm{supp}(\mathbf{x}) \bigcup \mathrm{supp}(\mathbf{v}-\mathbf{x}) \bigcup \left(\bigcup_{j=1}^{\beta_4} \mathrm{supp}(\mathbf{u}_j - \mathbf{x}) \right). \qquad (18)$$

Consider the following cases, counting as before the contributions of each case.

1) $\mathrm{wt}(\mathbf{y}) = 1$. We have $c_{2,1} = 2$.

2) $\mathrm{wt}(\mathbf{y}) = 2$. This case gives $a_{2,2} = 2(q-2)$.

3) $\mathbf{y} = \mathbf{v}$. Since H_3 ia a Steiner system, this happens only once. Hence, $b_{2,3} = 1$.

4) $\mathrm{supp}(\mathbf{y}) = \mathrm{supp}(\mathbf{v})$, $\mathbf{y} \neq \mathbf{v}$. Let the vector $\mathbf{x}_1$ of weight two be obtained from $\mathbf{y}$ by changing the first nonzero, say ℓ^*-th position of $\mathbf{x}$ to zero. For this $\mathbf{x}_1$ there is one vector, say $\mathbf{v}_1$ from H_3, covering $\mathbf{x}_1$. Also there are exactly β_4 codewords, say $\mathbf{w}_j$, $j = 1, ..., \beta_4$ from C_4, covering $\mathbf{x}_1$. All these vectors $\mathbf{x}_1$, $\mathbf{v}_1$, and $\mathbf{w}_j$, define the partition of set J, as in (18). Hence the ℓ^*-th element of J should be either in $\mathrm{supp}(\mathbf{v}_1)$ or in $\mathrm{supp}(\mathbf{w}_j - \mathbf{x}_1)$ for some j (it can not be in $\mathrm{supp}(\mathbf{x}_1)$ by the choice of $\mathbf{x}_1$). But both vectors $\mathbf{v}$ and $\mathbf{v}_1$ belong to H_3. Hence $d(\mathbf{v}, \mathbf{v}_1) = 3$ and $\mathbf{v}_1$ should have the zero element in its ℓ^*-th position.

Therefore, one of the vectors $\mathbf{w}_j$ will cover the ℓ^*-th position, which implies that $\mathbf{y}$ belongs to $C(2)$. This gives $a_{2,4} = q - 2$.

5) $\mathrm{supp}(\mathbf{y}) \subset \mathrm{supp}(\mathbf{u}_j)$, $\mathbf{y}$ is covered by $\mathbf{u}_j$. Since there are exactly $\beta_4 = (n-3)/2$ codewords $\mathbf{u}_j$ covering $\mathbf{y}$ we have clearly $c_{2,5} = n - 3$.

6) $\mathrm{supp}(\mathbf{y}) \subset \mathrm{supp}(\mathbf{u}_j)$, $\mathbf{y}$ is not covered by $\mathbf{u}_j$. In this case $\mathbf{y} \in C(2)$. Thus, we obtain $a_{2,6} = (q-2)(n-3)$.

Summing up contributions of all these cases, we obtain the expressions above for a_2, b_2 and c_2.

(ii) We have to find the parameters α_i, $i = 0, 1, 2, 3$. Since $d > \rho$ we have that $\alpha_0 = 1$. We can find easily α_3 since for any word $\mathbf{v}$ from H_3' we have $d(\mathbf{v}, C) = 3$, i.e. $\mathbf{v}$ is in $C(\rho) = C(3)$:

$$\alpha_3 \;=\; \frac{q-1}{|H_3'|} \;=\; \frac{6}{(q-1)n(n-1)}.$$

Now assume that $\mathbf{x} = (1, 0, 0, ..., 0)$. From (1) we have

$$\alpha_1 \cdot f_1(\mathbf{x}) + \alpha_3 \cdot f_3(\mathbf{x}) \;=\; 1. \tag{19}$$

It is clear that $f_1(\mathbf{x}) = 1$. Since the set C_4 is a 2-design, namely $T(n, 4, 2, \beta_4)_q$, we obtain that

$$f_3(\mathbf{x}) \;=\; \frac{1}{3} \cdot (n-1)(q-1)\beta_4 \;=\; \frac{1}{6}(q-1)(n-1)(n-3).$$

Using these formulas for $f_i(\mathbf{x})$ we find α_1:

$$\alpha_1 \;=\; (1 \;-\; \alpha_3 f_3(\mathbf{x})) \;=\; \frac{3}{n}.$$

which gives the expression for α_1 of the theorem.

Since we know that C is uniformly packed in the wide sense (since it is completely regular), we can find easily α_2 from the packing conditions (see [1]), namely

$$\sum_{i=0}^{3} \alpha_i (q-1)^i \binom{n}{i} = \frac{q^n}{|C|}.$$

$\triangle$

Now we prove one more theorem which gives for the case $q = 2^s$, $s = 2, 3, ...$ a new completely regular code of length $n = q + 1$ with $\rho = 3$, which is a subcode of a q-ary perfect code of such length. This case is connected with MDS codes, i.e. $(n, N, d)_q$ codes with cardinality $N = q^{n-d+1}$ (see [9]).

Theorem 5 *Let $q = 2^s \geq 4$ where $s = 2, 3...$. Let H be a q-ary perfect Hamming $[q+1, q-1, 3]_q$-code, i.e. H is also an MDS code. Then:*
(i) There is the $[q+1, q-2, 4]_q$ code C, which is a subcode of H.
(ii) C is a completely regular code with covering radius $\rho = 3$ and with intersection array

$$(q^2 - 1, q(q-1), 1; 1, q, q^2 - 1). \tag{20}$$

(iii) C is uniformly packed in the wide sense with parameters α_i:

$$\alpha_0 = 1,\ \alpha_1 = \frac{3}{q+1},\ \alpha_2 = \frac{6}{q(q+1)},\ \alpha_3 = \frac{6}{q(q^2-1)}. \tag{21}$$

Proof. Assume that $q = 2^s$, where $s = 2, 3, ...$ For this case we know that there exists the linear Hamming code $H = [q+1, q-1, 3]$ which it is also an MDS code. Let $\xi_0 = 0$, $\xi_1 = 1$, ξ_2, $\cdots$, ξ_{q-1} be the elements of $GF(q)$. Then the parity check matrix for H is

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 0 \\ 1 & \xi_2 & \cdots & \xi_{q-1} & 0 & 1 \end{pmatrix}$$

and

$$H_c = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 0 \\ 1 & \xi_2 & \cdots & \xi_{q-1} & 0 & 1 \\ 1 & \xi_2^2 & \cdots & \xi_{q-1}^2 & 0 & 0 \end{pmatrix}$$

is the parity check matrix for the linear subcode C of H with parameters $[q+1, q-2, 4]$. Code C is an MDS code of minimum distance 4 because any three columns in H_c are linearly independent. Note that any three of the first columns in H_c is a Vandermonde matrix. Also, given three columns including one or both of the last two columns, we can compute the determinant of these three columns and get always a nonzero result, since all the ξ_j^2 are different.

Now any MDS code $[n, k, d]_q$ has (see Section 11.4 in [9])

$$(q-1)\binom{n}{d}$$

codewords of minimum weight d. For our case we obtain, that C has cardinality

$$|C_4| \;=\; \frac{n(n-1)}{12}(q-1)^2\left(\frac{1}{2}(n-3)\right). \tag{22}$$

It is known that any MDS code is distance invariant (see [9]). Hence from Theorem 4 we have only to show that C_4 is a q-ary 2-design $T(q+1, 4, 2, \beta_4)$ with $\beta_4 = (q-2)/2$. From (22) it follows that in average each vector $\mathbf{x} \in \mathbb{F}_q^n$ of weight two is covered by β_4 codewords from C_4. But this value is an upper bound for this number. Indeed, recall that H_3 is a q-ary Steiner system. This means that $\mathbf{x}$ is covered by the unique codeword from H_3. Since C is a subcode of H, surely the words from C_4 are at distance 3 from H_3. So, for any $\mathbf{x} \in \mathbb{F}_q^n$ of weight 2 there is a unique position, say $j = j(\mathbf{x})$ which can not be covered by any codewords from C_4 covering $\mathbf{x}$. But this exactly means that β_4 can not be more than $(n-3)/2 = (q-2)/2$ (the vector $\mathbf{x}$ of weight two is covered by codewords of weight four). Now the statements of theorem follow from Theorem 4. $\triangle$

In fact, any q-ary perfect code of length n with $d = 3$ gives a completely regular code of length $n-1$ and covering radius $\rho = 2$.

Theorem 6 *Let H be a q-ary perfect $(n, N, 3)$ code where q is any natural number. Denote by $H^{(s)}$ the $(n-1, N/q, 3)$ code, obtained from H by shortening on zero element in the first position. Then:*
(i) $H^{(s)}$ is completely regular with $\rho = 2$ and with intersection array $((n-1)(q-1), q-1; 1, (n-1)(q-1))$.
(ii) $H^{(s)}$ is uniformly packed in the wide sense with parameters α_i:

$$\alpha_0 = 1, \quad \alpha_1 = \frac{2}{n-1}, \quad \alpha_2 = \frac{2}{(q-1)(n-1)}.$$

Proof. Since $d = 3$ and $\rho = 2$ we have immediately that

$$a_0 = 0, \; b_0 = (n-1)(q-1), \;\; \text{and} \;\; a_2 = 0, \; c_2 = (n-1)(q-1).$$

Thus, we have to find only a_1, b_1 and c_1. Let $\mathbf{x}$ be any vector of weight one, say, $\mathbf{x} = (1, 0, ..., 0)$ and let $\mathbf{y} \in W(\mathbf{x})$. We have to consider four cases.
1) $\mathrm{wt}(\mathbf{y}) = 0$. For this case we have $c_{1,1} = 1$.
2) $\mathrm{wt}(\mathbf{y}) = 1$. We have clearly $a_{1,2} = q-2$.
3) $\mathrm{wt}(\mathbf{y}) = 2$ and $\mathbf{y} \in H^{(s)}(2)_2$. Since H_3 is a Steiner system $S(n, 3, 2)_q$ this happens exactly $q-1$ times. Indeed, any vector $\mathbf{z} = (\gamma | 1, 0, ..., 0) \in F^n$ (where $\gamma \neq 0$) is covered by exactly one codeword, say $\mathbf{v}'_\gamma$ from H_3. For the code H this means that the vector $\mathbf{x} = (1, 0, ..., 0)$ is covered by exactly $q-1$ vectors $\mathbf{v}_\gamma$ obtained from $\mathbf{v}'_\gamma$ removing the first position. This gives

$b_{1,3} = q - 1$.

4) $\mathrm{wt}(\mathbf{y}) = 2$ and $\mathbf{y} \in H^{(s)}(1)$. We have to show, that the condition $\mathbf{y} \notin H^{(s)}(2)_2$ implies $\mathbf{y} \in H^{(s)}(1)$. Indeed, any vector $\mathbf{y}$ of weight two is covered by some codeword from H_3. But by the previous arguments (used for the case 3)), it can not be covered by any such word, having first nonzero position (we mentioned all such $q-1$ words $\mathbf{v}'_\gamma$). Hence these vectors $\mathbf{y}$ will be covered by vectors from H_3 having first zero position. But such vectors are codewords of new code $H^{(s)}$. Therefore, we deduce that $a_{1,4} = (n-1)(q-1) - 2q + 2$.

Summing up our results we obtain the expressions for the numbers a_1, b_1 and c_1. This gives (i). The second part (ii) follows similarly to the previous cases, since we know already that $H^{(s)}$ is completely regular and, hence, uniformly packed in the wide sense. $\triangle$

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