On non-antipodal binary completely regular codes ⋆

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Abstract

Binary non-antipodal completely regular codes are characterized. Using the result on nonexistence of nontrivial binary perfect codes, it is concluded that there are no unknown nontrivial non-antipodal completely regular binary codes with minimum distance $d \geq 3$. The only such codes are halves and punctered halves of known binary perfect codes. Thus, new such codes with covering radiuses $\rho = 2, 3, 6$ and $\rho = 7$ are obtained. In particular, a half of the binary Golay $[23, 12, 7]$-code is a new binary completely regular code with minimum distance $d = 8$ and covering radius $\rho = 7$. The punctured half of the Golay code is a new completely regular code with minimum distance $d = 7$ and covering radius $\rho = 6$. That new code with $d = 8$ disproves the known conjecture of Neumaier, that the extended binary Golay $[24, 12, 8]$-code is the only binary completely regular code with $d \geq 8$. Halves of binary perfect codes with Hamming parameters also provide an infinite family of new binary completely regular codes with $d = 4$ and $\rho = 3$. Puncturing of these codes also provide an infinite family of binary completely regular codes with $d = 3$

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and $\rho = 2$. Some of these new codes are also new completely transitive codes. Of course, all these new codes are new uniformly packed codes in the wide sense.

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1 Introduction

Let $\mathbb{F}^n$ be the n-dimensional vector space over $F = GF(2)$. The *Hamming weight*, $\mathrm{wt}(\mathbf{v})$, of a vector $\mathbf{v} \in \mathbb{F}^n$ is the number of its nonzero coordinates. The *Hamming distance* between two vectors $\mathbf{v}, \mathbf{u} \in \mathbb{F}^n$ is $d(\mathbf{v}, \mathbf{u}) = \mathrm{wt}(\mathbf{v} + \mathbf{u})$. A *(binary)* (n, N, d)*-code* C is a subset of $\mathbb{F}^n$ where n is the *length*, d is the *minimum distance*, and $N = |C|$ is the *cardinality* of C. Given any vector $\mathbf{v} \in \mathbb{F}^n$, its *distance to the code* C is

$$d(\mathbf{v}, C) = \min_{\mathbf{x} \in C} \{d(\mathbf{v}, \mathbf{x})\}$$

and the *covering radius* of the code C is

$$\rho(C) \; = \; \rho \; = \; \max_{\mathbf{v} \in \mathbb{F}^n} \{d(\mathbf{v}, C)\}$$

Given two sets $X, Y \subset \mathbb{F}^n$, define their minimum distance $d(X, Y) = \min\{d(\mathbf{x}, \mathbf{y}) : \; \mathbf{x} \in X, \; \mathbf{y} \in Y\}$. We write $X + \mathbf{x}$ instead of $X + \{\mathbf{x}\}$. For a given vector $\mathbf{x} \in \mathbb{F}^n$ let $\bar{\mathbf{x}}$ be the *complementary* vector, i.e. $d(\mathbf{x}, \bar{\mathbf{x}}) = n$. For a given set $X \subset \mathbb{F}^n$ define the complementary set $\bar{X} = \{\bar{\mathbf{x}} : \; \mathbf{x} \in X\}$. We write $\mathbf{1}$ (respectively $\mathbf{0}$) for the all one (respectively, all zero) vector in $\mathbb{F}^n$.

For a given code C with covering radius $\rho = \rho(C)$ define

$$C(i) \; = \; \{\mathbf{x} \in \mathbb{F}^n : \; d(\mathbf{x}, C) = i\}, \;\; i = 1, 2, ..., \rho.$$

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We assume that the code C always contains the zero vector $\mathbf{0}$, unless stated otherwise. Let $D = C + \mathbf{x}$ be a *shift* of C. The *weight* $\mathrm{wt}(D)$ of D is the minimum weight of the codewords of D. For an arbitrary shift D of weight $i = \mathrm{wt}(D)$ denote by $\mu(D) = (\mu_0(D), \mu_1(D), ..., \mu_n(D))$ its weight distribution ($\mu_i(D)$ is the number of words of D of weight i). So $\mu(C) = (\mu_0(C), ..., \mu_n(C))$ is the weight distribution of C. If this vector $\mu(C)$ is the same for any shift of C by a codeword, then C is *distance invariant*. Denote by C_j (respectively, D_j, and $C(i)_j$) the subset of C (respectively, of D and $C(i)$), formed by all words of the weight j. In our terminology $\mu_i(D) = |D_i|$.

A (n, N, d) code C with minimum distance $d = 2e + 1$ we *extend* to $(n + 1, N, d + 1)$ code C^*, adding one overall parity checking symbol to codewords of C, and *puncture* to $(n - 1, N, d - 1)$ code $C^{(1)}$, deleting any one position of codewords of C

Definition 1 *A code C is a completely regular if, for all $l \geq 0$, every vector $x \in C(l)$ has the same number c_l of neighbors in $C(l - 1)$ and the same number b_l of neighbors in $C(l + 1)$. Also, define $a_l = n - b_l - c_l$ and note that $c_0 = b_\rho = 0$.*

Define by $\{b_0, \ldots, b_{\rho-1}; c_1, \ldots, c_\rho\}$ the *intersection array* of C and by L the *intersection matrix* of C:

$$L = \begin{pmatrix} a_0 & b_0 & 0 & 0 & \cdots & 0 \\ c_1 & a_1 & b_1 & \cdots & 0 & 0 \\ 0 & c_2 & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \vdots & \ddots & \ddots & b_{\rho-1} \\ 0 & 0 & \cdots & \cdots & c_\rho & a_\rho \end{pmatrix}.$$

For a binary code C let $\mathrm{Perm}(C)$ be its permutation stabilizer group. For any $\theta \in \mathrm{Perm}(C)$ and any shift $D = C + \mathbf{x}$ of C define the action of θ to D as: $\theta(D) = C + \theta(\mathbf{x})$.

Definition 2 *Let C be a binary additive code with covering radius ρ. The code C is called* completely transitive*, if the set $\{C + \mathbf{x} : \mathbf{x} \in \mathbb{F}^n\}$ of all different shifts of C is partitioned under action of* $\mathrm{Perm}(C)$ *exactly into $\rho + 1$ orbits.*

Since two shifts in the same orbit should have the same weight distribution, it is clear, that any completely transitive code is completely regular.

It has been conjectured for a long time that if C is a completely regular code and $|C| > 2$, then $e \leq 3$. Moreover, in [11] it is conjectured that the only completely regular code C with $|C| > 2$ and $d \geq 8$ is the extended binary Golay $[24, 12, 8]$-code with $\rho = 4$. As we know from [15,17] for $\rho = e$ and [16] (see also [14,8]) for $\rho = e + 1$, any such nontrivial unknown code should have a covering radius $\rho \geq e + 2$. For the special case of completely regular codes, for linear completely transitive codes [12], the problem of existence is solved: we [2,3] proved that for $e \geq 4$ such nontrivial codes do not exist.

In this paper we give a complete characterization of binary nontrivial, non-antipodal, completely regular codes with distance $d \geq 3$. The only such codes are formed by halves of binary perfect codes. In particular, a half of the binary Golay $[23, 12, 7]$ code is a new non-antipodal completely regular $[23, 11, 8]$ code with covering radius $\rho = 7$ and intersection array

$$(23, 22, 21, 20, 3, 2, 1; 1, 2, 3, 20, 21, 22, 23).$$

This result implies that the conjecture of Neumaier [11] is not valid. The punctured half of the Golay code is a new non-antipodal completely regular $[22, 11, 7]$ code with covering radius $\rho = 6$ and intersection array

$$(22, 21, 20, 3, 2, 1; 1, 2, 3, 20, 21, 22).$$

Halves of binary perfect $(n, N, 3)$ codes also give a new infinite family of completely regular codes with $d = 4$, $\rho = 3$ and intersection array $(n, n-1, 1; 1, n-1, n)$. The punctured halves of binary perfect $(n, N, 3)$ codes are uniformly packed in the narrow sense [14], and therefore are completely regular with $d = 3$, $\rho = 3$ and intersection array $(n, 1; 1, n)$. The same results are valid for q-ary perfect codes, under certain conditions on original codes. In particular, from the ternary Golay code we obtain new ternary completely regular code with minimum distance 6, with covering radius 5 and with intersection array $(22, 20, 18, 2, 1; 1, 2, 9, 20, 22)$. New completely regular codes are considered in separate paper [4].

The present paper is organized as follows. In Section 2 we give some preliminary results concerning completely regular codes. In Section 3 we prove that the covering set $C(\rho)$ of non-antipodal completely regular binary code C is its shift by $\mathbf{1}$. This permits to us to lower and upper bound the covering radius of non-antipodal completely regular codes. In Section 4 we prove that the only non-antipodal completely regular codes are formed either by even (or odd) codewords of any binary perfect codes, or the codes, obtained by puncturing these codes.

2 Preliminary results

We give some definitions, and results which we will need later.

Definition 3 *Let C be any binary code of length n and let ρ be its covering radius. We say that such a code is* uniformly packed *in the wide sense, i.e. in the sense of [1], if there exist rational numbers $\alpha_0, \ldots, \alpha_\rho$ such that for any $\mathbf{v} \in \mathbb{F}^n$*

$$\sum_{k=0}^{\rho} \alpha_k f_k(\mathbf{v}) = 1 , \qquad (1)$$

where $f_k(\mathbf{v})$ is the number of codewords at distance k from $\mathbf{v}$. We say that *such a code is* strongly uniformly packed, *or uniformly packed in the sense of [14], if* $\rho = e + 1$ *and* $\alpha_e = \alpha_{e+1}$, *where* $e = \lfloor (d-1)/2 \rfloor$.

The *support* of $\mathbf{v} \in \mathbb{F}^n$, $\mathbf{v} = (v_1, \ldots, v_n)$ is $supp(\mathbf{v}) = \{ \ \ell \mid v_\ell \neq 0 \ \}$. Say that a vector $\mathbf{v}$ *covers* a vector $\mathbf{z}$ if $supp(\mathbf{z}) \subseteq supp(\mathbf{v})$.

Definition 4 *A* t-design $T(n, w, t, \beta)$ *is a set of binary vectors of length* n *and weight* w *such that for any binary vector* $\mathbf{z}$ *of weight* t, $1 \leq t \leq w$, *there are precisely* β *vectors* $\mathbf{v}_i$, $i = 1, ..., \beta$, *of* $T(n, w, t, \beta)$ *each of them covering* $\mathbf{z}$. *If* $\beta = 1$ *the design* $T(n, w, t, 1)$ *is a Steiner system* $S(n, w, t)$.

Definition 5 *Say that a binary code* C *is* even *(respectively,* odd*) if all its codewords have even (respectively,* odd*) weights.*

The next fact follows from the definition of completely regular code.

Lemma 6 *Let* C *be a completely regular code with minimum distance* d *and with zero codeword. Then any nonempty set* C_j, $d \leq j \leq n$, *is a* t*-design, where* $t = e$, *if* $d = 2e + 1$ *and* $t = e + 1$, *if* $d = 2e + 2$.

Lemma 7 *[11] If* C *is completely regular with covering radius* ρ, *then* $C(\rho)$ *is also completely regular, with reversed intersection array.*

Definition 8 *The code* C *is called* antipodal, *if for any* $\mathbf{c} \in C$ *the complementary vector* $\bar{\mathbf{c}} = \mathbf{c} + \mathbf{1}$ *is also a codeword of* C.

It is clear that a distance invariant code C, containing $\mathbf{0}$, is antipodal if it contains $\mathbf{1}$.

Lemma 9 *Let* C *be any binary code. Then* C *and* $C(\rho)$ *are antipodal or not simultaneously.*

PROOF. Let C be any binary code, and let $C(\rho)$ be the corresponding covering set of C. Assume that C is antipodal. To see that $C(\rho)$ is antipodal we take $\mathbf{v} \in C(\rho)$ and prove that $\mathbf{1} + \mathbf{v} \in C(\rho)$. In order to do this we observe that $d(\mathbf{1} + \mathbf{v}, C) = \rho$, since

$$d(\mathbf{1} + \mathbf{v}, C) = d(\mathbf{v}, \mathbf{1} + C) = d(\mathbf{v}, C) = \rho.$$

The statement follows now since the antipodality of $C(\rho)$ implies the antipodality of C by reversing of C and $C(\rho)$. $\square$

3 On covering radius of non-antipodal binary completely regular codes

The natural question is: *does any completely regular code contain the vector* $\mathbf{1}$?

Theorem 10 *Let C be a completely regular code with covering radius ρ, with minimum distance $d \geq 3$. If $\mathbf{0} \in C$, but $\mathbf{1} \notin C$, then $\mathbf{1} \in C(\rho)$ and $C + \mathbf{1} = C(\rho)$. Furthermore*

$$\rho \;\geq\; \begin{cases} 2e, & \textit{if } \; d = 2e+1, \\ 2e+1 & \textit{if } \; d = 2e+2. \end{cases}$$

PROOF. Let C be a completely regular code and let $\mathbf{1} \notin C$. First we prove that $\mathbf{1} \in C(\rho)$. In contrary, assume that $\mathbf{1} \notin C(\rho)$. Consider the subset C_w of C of the largest weight w and the subset $C(\rho)_v$ of $C(\rho)$ of the largest weight v. As C and $C(\rho)$ do not contain $\mathbf{1}$, we have clearly: $1 \leq n - w \leq \rho - 1$ and $1 \leq n - v \leq \rho - 1$.

Now we claim that

$$(n - w) \; + \; (n - v) \; = \; \rho. \tag{2}$$

Indeed, C is a completely regular code in the Hamming space $\mathbb{F}^n$, which is a metric association scheme [7]. In particular, this means that for any vector $\mathbf{x}$ from $\mathbb{F}^n$ there exist two vectors $\mathbf{c} \in C$ and $\mathbf{v} \in C(\rho)$ such that

$$d(\mathbf{c},\, \mathbf{x}) \;+\; d(\mathbf{x},\, \mathbf{v}) \;=\; \rho. \tag{3}$$

Taking the vector $\mathbf{1}$ as $\mathbf{x}$ we immediately obtain (2), since we have that $d(\mathbf{1},\, C_w) = n - w$ and $d(\mathbf{1},\, C(\rho)_v) = n - v$.

By Lemma 6, the set C_w is a t-design, say $T_1(n, w, t, \beta_1)$ with $t = e$, or $t = e+1$, where $e = \lfloor (d-1)/2 \rfloor$. By the condition of theorem $d \geq 2$ and, therefore, $t \geq 1$. By definition, $d(C,\, C(\rho)) = \rho$. Hence, these sets C_w and $C(\rho)_v$ are at distance ρ at least from each other. Consider the complementary sets: $S_1 = \bar{C}_w$ with vectors of weight $w' = n - w$ and $S_2 = \bar{C}(\rho)_v$ with vectors of weight $v' = n - v$, where $1 \leq w', v' \leq \rho - 1$. From (2) we deduce that

$$w' \;+\; v' \;=\; \rho. \tag{4}$$

But the set S_1 (which is complementary of C_w) is a t-design also [13], say $T(n, w', t, \alpha_1)$ with $t \geq 1$. Taking any word $\mathbf{z}$ from S_2 we can always find $\mathbf{x}$ from S_1 such that $|\mathrm{supp}(\mathbf{z}) \cap \mathrm{supp}(\mathbf{x})| \geq t$. Taking into account this last fact, we conclude that under the condition (4) these two sets S_1 and S_2 have the minimum distance $d(S_1, S_2) \leq \rho - 2$. Thus, we obtain a contradiction and $\mathbf{1}$ should belong to $C(\rho)$.

Now we claim that $C + \mathbf{1}$ belongs to $C(\rho)$. This comes from the fact that C is completely regular, and, therefore, the distance distribution is the same for all its codewords. And this distance distribution says that for any codeword $\mathbf{c} \in C$ the complementary vector $\bar{\mathbf{c}}$ belongs to $C(\rho)$. We conclude, therefore, that $C + \mathbf{1}$ is a subset of $C(\rho)$. But $C + \mathbf{1}$ is a shift of C of weight ρ, and any such shift has the same weight distribution. But there is only one vector $\mathbf{1}$ of weight n. So, we can have only one such shift. This means that $|C(\rho)| = |C|$

and, therefore,

$$C + \mathbf{1} \;=\; C(\rho). \tag{5}$$

This last property implies immediately limitations for the possible values of ρ. Indeed, since $\mathbf{1}$ belongs to $C(\rho)$ the set $C_{n-\rho}$ is not empty as well as the set $C(\rho)_\rho$, since (5). As $C_{n-\rho}$ is a t-design (Lemma 6) the set $C(\rho)_\rho$ is a t-design too [13], say $T_2(n,\rho,t,\beta_2)$. By (5) we deduce that $C(\rho)_\rho$ is a constant weight code with minimum distance $d(T_2) \geq 2e+2$. If $d = 2e+1$, we have $t = e$ (Lemma 6). This implies that $\rho \geq 2e$ if $\beta_2 = 1$ and $\rho \geq 2e+1$ if $\beta_2 > 1$. If $d = 2e+2$, we have $t = e+1$ (Lemma 6). This implies that $\rho \geq 2e+1$ if $\beta_2 = 1$ and $\rho \geq 2e+2$ if $\beta_2 > 1$. □

Lemma 11 *Let C and its (even or odd) extension C^* be completely regular codes of lengths n and $n+1$ and with covering radii ρ and $\rho+1$, respectively. Then C and C^* are antipodal or not simultaneously.*

PROOF. Let $\mathbf{1} \in C$. Assume, in contrary that $\mathbf{1} \notin C^*$. Then by Theorem 10, $\mathbf{1} \in C^*(\rho+1)$, and, therefore, $\mathbf{1} \in C(\rho)$, i.e. a contradiction. If C^* is antipodal, then clearly C is antipodal too. □

The next two theorems upper bound the covering radius of any non-antipodal nontrivial completely regular binary code.

Theorem 12 *Let C be a nontrivial non-antipodal completely regular code with covering radius ρ, with minimum distance $d = 2e+1 \geq 3$ and with zero word. Then $\rho = 2e$.*

PROOF. From Theorem 10 we have that $\rho \geq 2e$. Assume that $\rho \geq 2e+1$. By Lemma 6 the set C_d is a e-design, say $T_d(n, 2e+1, e, \lambda_d)$ and the set $C(\rho)_\rho$ (since it is a complementary [13] of $C_{n-\rho}$ which is also a e-design by Lemma 6) is an e-design too, say $T_\rho(n,\rho,e,\lambda_\rho)$.

Let J denote the coordinate set of C, i.e. $J = \{1, 2, ..., n\}$. For a given vector $\mathbf{x}$ of weight e define the following subsets of J: $J_x = supp(\mathbf{x})$, $J_d^{(1)}$ is the union of $\mathrm{supp}(\mathbf{c}) \setminus \mathrm{supp}(\mathbf{x})$ of all codewords $\mathbf{c}$ from C_d, covering $\mathbf{x}$, and $J_\rho^{(1)}$ is the union of $\mathrm{supp}(\mathbf{v}) \setminus \mathrm{supp}(\mathbf{x})$ of all words $\mathbf{v}$ from $C(\rho)_\rho$, covering $\mathbf{x}$. We note that $J_d^{(1)}$ and $J_\rho^{(1)}$ are disjoint, since any two words $\mathbf{c} \in C_d$ and $\mathbf{v} \in C(\rho)_\rho$ can not have more than e common nonzero positions, i.e.

$$\mathrm{supp}(\mathbf{c}) \cap \mathrm{supp}(\mathbf{v}) \;\leq\; e. \tag{6}$$

Now we claim that

$$J \;=\; J_x \bigcup J_d^{(1)} \bigcup J_\rho^{(1)}. \tag{7}$$

Indeed, it comes from the fact that C is completely regular, and any vector $\mathbf{z}$ of weight $e+1$ covering $\mathbf{x}$ should be covered either by some codeword $\mathbf{c}$ from C_d, or by some vector $\mathbf{v}$ from $C(\rho)_\rho$. If it is not covered by any codeword from C_d, then $\mathbf{z}$ is at distance $e+1$ from C. Therefore, it should be at distance $\rho-(e+1)$ from $C(\rho)$. This means that there is a vector $\mathbf{v} \in C(\rho)_\rho$ covering $\mathbf{z}$. If we assume that $\mathbf{z}$ is covered by some possible vector $\mathbf{u} \in C(\rho)_{\rho+1}$, we will have $d(\mathbf{z}, \mathbf{u}) = d(\mathbf{z}, C(\rho)) = \rho - e$, i.e. a contradiction with $d(\mathbf{z}, C) = e+1$.

Thus, any vector $\mathbf{x}$ of weight e induces a partition of the coordinate set J into three disjoint subsets $J_{\mathbf{x}}$, $J_d^{(1)}$ and $J_\rho^{(1)}$. Since C_d is $T_d(n, d, e, \lambda_d)$, for any such vector $\mathbf{x}$ of weight e we have

$$|J_d^{(1)}| \;=\; (e+1)\lambda_d. \tag{8}$$

Remark that we can not write such kind of expression for $J_\rho^{(1)}$, since we do not know the value of ρ and the minimum distance of $C(\rho)_\rho$.

Having these two equalities (7) and (8), it is easy to write out the intersection numbers (a_e, b_e, c_e) for any such vector $\mathbf{x}$ of weight e:

$$c_e = e,\; a_e = |J_d^{(1)}| = (e+1)\lambda_d,\; b_e = |J_\rho^{(1)}|. \tag{9}$$

Now we write out the numbers $(a_{\rho-e}, b_{\rho-e}, c_{\rho-e})$ which according to Lemma 7, should be reversed of (a_e, b_e, c_e). For a given fixed $\mathbf{v}^* \in C(\rho)_\rho$ of weight ρ let $\mathbf{y}$ be any vector of weight $\rho - e$ which is covered by $\mathbf{v}^*$. We have

$$c_{\rho-e} = \rho - e,\ a_{\rho-e} = |J_d^{(2)}|,\ b_{\rho-e} = e + |J_\rho^{(2)}|. \tag{10}$$

Here the set $J_d^{(2)}$ is the union of $\mathrm{supp}(\mathbf{c}) \setminus \mathrm{supp}(\mathbf{y})$ of all vectors $\mathbf{c}$ from C_d, having e nonzero positions in $\mathrm{supp}(\mathbf{y})$. The set $J_\rho^{(2)}$ is the rest of J:

$$J_\rho^{(2)} \;=\; J \setminus \left(\mathrm{supp}(\mathbf{v}^*) \bigcup J_d^{(2)}\right).$$

Now by Lemma 7 we have that $a_e = a_{\rho-e}$ and $b_e = c_{\rho-e}$. Taking into account (9) and (10), we obtain

$$a_e = |J_d^{(1)}| = a_{\rho-e} = |J_d^{(2)}| = (e+1)\lambda_d. \tag{11}$$

and $|J_\rho^{(1)}| = \rho - e$. From the last equality we deduce that $C(\rho)_\rho$ (which is $T_\rho(n, \rho, e, \lambda_\rho)$) is a Steiner system $S(n, \rho, e)$. This implies that

$$\frac{n-e+1}{\rho-e+1} \;=\; \lambda_S. \tag{12}$$

is integer (since $S(n, \rho, e)$ is also $(e-1)$-design $T(n, \rho, e-1, \lambda_S)$). Furthermore, existence of $S(n, \rho, e)$ for $e > 1$ implies existence of $S(n-e+2, \rho-e+2, 2)$ for which the Fisher inequality (see, for example, [9]) states that

$$|S(n-e+2, \rho-e+2, 2)| \geq n-e+2.$$

This can be written as: for $e > 1$

$$n \geq (\rho - e + 1)^2 + \rho. \tag{13}$$

Using (7) and (8), we obtain that $(n-\rho)/(e+1)$ is integer:

$$\lambda_d = \frac{n-\rho}{e+1}. \tag{14}$$

Now we want to upper bound ρ. For the case $e > 1$ we fix any $\mathbf{v}^* \in C(\rho)_\rho$, take any $\mathbf{x} \in \mathbb{F}^n$ of weight $e-1$, which is covered by $\mathbf{v}^*$, and define the following partitions of the set $J \setminus \mathrm{supp}(\mathbf{v}^*)$. The partition $S_1, S_2, ..., S_{\lambda_S - 1}$ is formed by $\mathrm{supp}(\mathbf{v}) \setminus \mathrm{supp}(\mathbf{x})$ of $\lambda_S - 1$ vectors $\mathbf{v} \neq \mathbf{v}^*$ from $C(\rho)_\rho$, which covers $\mathbf{x}$ (see (12)) and $(\rho - e + 1)$ partitions $L_1(i), L_2(i), ..., L_{\lambda_d}(i)$ (see (14)), where $i \in \mathrm{supp}(\mathbf{v}^*) \setminus \mathrm{supp}(\mathbf{x})$, formed by $\mathrm{supp}(\mathbf{b}(i)) \setminus \mathrm{supp}(\mathbf{x})$ of λ_d vectors $\mathbf{b}(i) \in C_d$ covering the set $\{i\} \cup \mathrm{supp}(\mathbf{x})$, which is a subset of $\mathrm{supp}(\mathbf{v}^*)$.

For $e = 1$ the Steiner system $S(n, \rho, 1)$ consists of $\lambda_S = n/\rho$ disjoint blocks. Hence we fix one of its blocks $\mathbf{v}^*$. Similarly, since $i \in \mathrm{supp}(\mathbf{v}^*)$, we obtain ρ different partitions $L_1(i), L_2(i), ..., L_{\lambda_d}(i)$, defined by i only.

By constructions we have the following properties of these partitions:
$(P.1)$ The partition $S_1, S_2, ..., S_{\lambda_S}$ and the partition $L_1(i), L_2(i), ..., L_{\lambda_d}(i)$ for any $i \in \mathrm{supp}(\mathbf{v}^*) \setminus \mathrm{supp}(\mathbf{x})$ are disjoint, i.e. for all $r, k, i, j, s,\ \ r \neq k,\ \ j \neq s$:

$$S_r \bigcap S_k = \emptyset, \text{ and } L_j(i) \bigcap L_s(i) = \emptyset.$$

$(P.2)$ For any $r, k, i, j, s,\ \ i \neq j$, we have that

$$|S_r \bigcap L_k(i)| \leq 1, \text{ and } |L_r(i) \bigcap L_s(j)| \leq 1.$$

$(P.3)$ For any vector $\mathbf{z}$ of weight two with $\mathrm{supp}(\mathbf{z}) \in J \setminus \mathrm{supp}(\mathbf{v}^*)$ there is the set, either S_j, or $L_k(i)$, containing $\mathrm{supp}(\mathbf{z})$.

The property $(P.1)$ we have by definition of partitions. The first inequality of $(P.2)$ follows from (6) and the second follows, since any two words of C_d have not more than e common nonzero positions. Now $(P.3)$ follows from the fact (which we already mentioned) that any vector $\mathbf{y}$ of weight $e+1$ covering $\mathbf{x}$ should be covered either by some vector from C_d, or by some vector from $C(\rho)_\rho$ (see (7)).

Now count by the two different ways the number of all vectors $\mathbf{z}$ of weight two with $\mathrm{supp}(\mathbf{z}) \in J \setminus \mathrm{supp}(\mathbf{v}^*)$. By $(P.3)$ this number is equal to

$$(\rho - e + 1)\binom{e+1}{2}\lambda_d \; + \; \binom{\rho - e + 1}{2}(\lambda_S - 1).$$

By definition this number is equal to

$$\binom{n-\rho}{2}.$$

Using (12) and (14), we deduce from the equality of these two numbers that

$$n \; = \; (e+1)(\rho - e + 1) + \rho. \tag{15}$$

Now, for the case $e > 1$, from (13) and (15) we obtain that $\rho \leq 2e$, which combining with Theorem 10 implies that $\rho = 2e$.

For the case $e = 1$ this expression (15) reduces to the following one:

$$n \; = \; 3\rho, \tag{16}$$

or $\lambda_d = \rho$, if we take into account (14). Since we do not have (13) for $e = 1$, this is not enough in order to upper bound ρ properly. To do it we extend the sets $C(\rho)_{\rho+1}$ and $C(\rho)_{\rho+1}$.

Let $\mathbf{x}$ be any vector of weight e, and let $\mathbf{z}$ be a vector of weight $e+1$ covering $\mathbf{x}$. Assume that $\mathbf{z}$ has one nonzero position on $J_d^{(1)}$, i.e. $\mathbf{z}$ is covered by some codeword from C_d. Denote by ξ_ρ the number of vectors from $C(\rho)_{\rho+1}$ which cover $\mathbf{z}$. Since $d(\mathbf{z}, C) = e$, there exist some vector $\mathbf{u}$ from $C(\rho)_{\rho+1}$, which covers $\mathbf{z}$, implying that $d(\mathbf{z}, C(\rho)) = \rho - e$ how it should be. But $d(\mathbf{x}, C(\rho)) = \rho - e$ also, and there is exactly one $\mathbf{v}$ covering $\mathbf{x}$. We conclude that there is exactly one $\mathbf{u}$ from $C(\rho)_{\rho+1}$ covering $\mathbf{z}$. Thus, any vector $\mathbf{z} \in \mathbb{F}^n$ of weight $e+1$ is covered by exactly one vector from $C(\rho)_{\rho+1}$ and any vector $\mathbf{x} \in \mathbb{F}^n$ of weight e is covered by exactly one vector from $C(\rho)_\rho$. We deduce that adding one position with 0 to all vectors from $C(\rho)_{\rho+1}$ and one position with 1 to all vectors from $C(\rho)_\rho$ we get, respectively, $C^*(\rho)_{\rho+1}$ and $C^*(\rho)_\rho$ and the union $C^*(\rho)_{\rho+1} \cup C^*(\rho)_\rho$ results in a Steiner system $S(n+1, \rho+1, e+1)$.

Now we apply the Fisher inequality to this Steiner system, which is a 2-design for $e = 1$, i.e.:

$$|S(n+1, \rho+1, 2)| \;\geq\; (n+1).$$

This reduces to the inequality

$$n \;\geq\; (\rho+1)\rho.$$

Combining with (16) we deduce that $\rho \leq 2$ and from Theorem 10, for the case $e = 1$, we obtain $\rho = 2$. The theorem is proved. □ □

Now we consider non-antipodal codes with even distance.

Theorem 13 *Let C be a nontrivial non-antipodal completely regular code with covering radius ρ, with minimum distance $d \geq 4$ and with zero word. If $d = 2e+2 \geq 4$, then $\rho = 2e+1$. Furthermore, $C(\rho)_\rho$ is a Steiner system $S(n, 2e+1, e+1)$.*

PROOF. Assume in contrary that $\rho \geq 2e+2$. From Theorem 10 we have that $\rho \geq 2e+1$. Note that C_d is $(e+1)$-design, say $T_d(n, d, e+1, \lambda_d)$, and also $C(\rho)_\rho$ is $T_\rho(n, \rho, e+1, \lambda_\rho)$. Compute the intersection numbers (a_i, b_i, c_i) for $i = e+1$ and for $i = \rho - e - 1$.

For a fixed vector $\mathbf{x}$ of weight $e+1$ denote $J_{\mathbf{x}} = \mathrm{supp}(\mathbf{x})$. Define three subsets of $J \setminus J_{\mathbf{x}}$. The set $J_d^{(1)}$ is formed by the supports of λ_d codewords $\mathbf{c}$ from C_d covering $\mathbf{x}$, the set $J_\rho^{(1)}$ is formed by the supports of λ_ρ vectors $\mathbf{v}$ from $C(\rho)_\rho$ covering $\mathbf{x}$, and the set $J_{d+1}^{(1)}$:

$$J_{d+1}^{(1)} = J \setminus \left(J_{\mathbf{x}} \bigcup J_d^{(1)} \bigcup J_\rho^{(1)}\right).$$

Since any $\mathbf{c} \in C_d$ and $\mathbf{v} \in C(\rho)_\rho$ can have not more than $e+1$ common nonzero positions, i.e.

$$|\mathrm{supp}(\mathbf{c}) \bigcap \mathrm{supp}(\mathbf{v}) \quad \leq \; e+1, \tag{17}$$

we conclude that $J_d^{(1)}$ and $J_\rho^{(1)}$ are disjoint. Therefore, J is partitioned into four disjoint subsets $J_{\mathbf{x}}$, $J_d^{(1)}$, $J_\rho^{(1)}$ and $J_{d+1}^{(1)}$. Denote such a partition by $P(\mathbf{x})$, since it is uniquely defined by $\mathbf{x}$. Now we claim that $J_{d+1}^{(1)}$ is formed by all λ_{d+1} codewords from C_{d+1} covering $\mathbf{x}$. First, note that any such $\mathbf{b} \in C_{d+1}$ (if it exists), which covers $\mathbf{x}$, does not have any nonzero positions on $J_d^{(1)}$ and $J_\rho^{(1)}$ (indeed, C is a code with $d = 2e+2$ and $\rho \geq 2e+2$). To see that any element of $J_{d+1}^{(1)}$ is contained in some $\mathbf{b} \in C_{d+1}$, assume that it is not the case. Let a vector $\mathbf{y}$ of weight $e+2$ cover $\mathbf{x}$ and be not covered by any word from $J_d^{(1)}$, $J_\rho^{(1)}$ or $J_{d+1}^{(1)}$. This means that $\mathbf{y}$ is at distance $e+2$ from zero word, C_d, and C_{d+2} (if it is nonempty), at distance $e+3$ from C_{d+1}, at distance $\rho - e$ from $C(\rho)_\rho$, and at distance $\rho - e - 1$ from $C(\rho)_{\rho+1}$ (if it is nonempty also). Hence, $\mathbf{y}$ has distance $e+2$ from C and distance $\rho - e - 1$ from $C(\rho)$, which is impossible. The only possibility is that $\mathbf{y}$ is covered by some word from C_{d+1}. We conclude also that C_{d+1} and $C(\rho)_{\rho+1}$ are empty or not simultaneously.

Since C_d is an $(e+1)$-design, we know the cardinality of $J_d^{(1)}$. Indeed, the λ_d codewords $\mathbf{c}$ from C_d, which cover $\mathbf{x}$, have disjoint supports on $J_d^{(1)}$. Taking into account that the sets $J_d^{(1)}$, $J_\rho^{(1)}$ and $J_{d+1}^{(1)}$ are disjoint, we conclude that

$$|J_d^{(1)}| = (e+1)\lambda_d. \tag{18}$$

Using this partition $P(\mathbf{x})$, we have for the case $i = e+1$:

$$a_{e+1} = |J_{d+1}^{(1)}|, \; b_{e+1} = |J_\rho^{(1)}|, \; c_{e+1} = e+1+|J_d^{(1)}| = (e+1)(\lambda_d+1). \tag{19}$$

Now we fix any $\mathbf{v}^* \in C(\rho)_\rho$ and any $\mathbf{y} \in \mathbb{F}^n$ of weight $\rho - e - 1$ which is covered by $\mathbf{v}^*$. We define on $J \setminus \mathrm{supp}(\mathbf{v}^*)$ three sets $J_d^{(2)}$, $J_\rho^{(2)}$, and $J_{d+1}^{(2)}$. The set $J_d^{(2)}$

is formed by all vectors $\mathbf{c}$ from C_d such that

$$|\text{supp}(\mathbf{c}) \bigcap \text{supp}(\mathbf{y})| = (e+1). \tag{20}$$

The set $J_\rho^{(2)}$ is formed by supp($\mathbf{v}$) of words $\mathbf{v}$ from $C(\rho)_\rho$ such that

$$|\text{supp}(\mathbf{v}) \bigcap \text{supp}(\mathbf{y})| \ = \ \rho - e - 1. \tag{21}$$

The set $J_{d+1}^{(2)}$ is the rest of $J \setminus \text{supp}(\mathbf{v}^*)$:

$$J_{d+1}^{(2)} \ = \ J \setminus \left(\text{supp}(\mathbf{v}^*) \bigcup J_d^{(2)} \bigcup J_\rho^{(2)}\right).$$

The sets $J_d^{(2)}$ and $J_\rho^{(2)}$ are disjoint. Indeed, if we assume that there is an element $i \in J$ such that

$$i \in J_d^{(2)} \bigcap J_\rho^{(2)},$$

then we obtain two vectors $\mathbf{c} \in C_d$ and $\mathbf{v} \in C(\rho)_\rho$, with $e+2$ common nonzero positions, which is impossible. Denote such a partition by $P(\mathbf{v}^*, \mathbf{y})$.

Having these sets, we have for the case $i = \rho - (e+1)$:

$$a_{\rho-e-1} = |J_{d+1}^{(2)}|, \ b_{\rho-e-1} = e + 1 + |J_\rho^{(2)}|, \ c_{\rho-e-1} = \rho - e - 1 + |J_d^{(2)}|. \tag{22}$$

By Lemma 7 we should have $a_{e+1} = a_{\rho-e-1}$, $b_{e+1} = c_{\rho-e-1}$ and $c_{e+1} = b_{\rho-e-1}$, which means (using (19) and (22)), that

$$|J_{d+1}^{(1)}| \ = \ |J_{d+1}^{(2)}|, \ |J_d^{(1)}| \ = \ |J_\rho^{(2)}|, \ \text{ and } \ |J_d^{(2)}| + \rho - e - 1 \ = \ |J_\rho^{(1)}|. \tag{23}$$

Denote $\ell_d = |J_d^{(1)}|$ and $\ell_\rho = |J_\rho^{(1)}|$. ¿From (18) we have that $\ell_d \ = \ \lambda_d(e+1)$. Denote by $W(x)$ the sphere of radius one with center at $\mathbf{x}$. For $\mathbf{z} \in W(\mathbf{x})$ with position $i \in \text{supp}(\mathbf{z})$ denote by $\xi_d(i)$ (respectively, by $\xi_\rho(i)$) the number of vectors from C_d (respectively, from $C(\rho)_\rho$) covering $\mathbf{z}$. From (18) we deduce that $\xi_d(i) = 1$ for any $i \in J_d^{(1)}$.

Our first step is to obtain the exact expressions for λ_ρ, $\xi_\rho(i)$, and ℓ_ρ. In all lemmas below the conditions of Theorem 13 are satisfied and $\rho \geq 2e + 2$.

Lemma 14 *We have that*

$$\lambda_\rho = \binom{\rho - e - 1}{e + 1} \cdot \lambda_d. \tag{24}$$

and for any $i \in J_\rho^{(1)}$

$$\xi_\rho(i) \ = \ \xi_\rho \ = \ \frac{\rho - e - 1}{\ell_\rho} \cdot \lambda_\rho \ = \ \binom{\rho - e - 2}{e} \cdot \lambda_d. \tag{25}$$

PROOF. The partition $P(\mathbf{v}^*, \mathbf{y})$ becomes $P(\mathbf{v}^* + \mathbf{y})$ if we shift it by the vector $\mathbf{v}^*$. Under this shift the roles of C and $C(\rho)$ interchange (indeed, C is the shift of $C(\rho)$). The vectors with supports on $J_d^{(2)} \cup \mathrm{supp}(\mathbf{y})$ will be the vectors from $J_\rho^{(1)}$. The vectors $\mathbf{c}_i$ from $J_d^{(2)} \cup \mathrm{supp}(\mathbf{y})$ corresponds to the vectors $\mathbf{u}_j$ of weight $\rho - e - 1$ on $J_\rho^{(1)}$. They are exactly the vectors $\mathbf{c}_i$, having $e + 1$ nonzero positions on $\mathrm{supp}(\mathbf{y})$. But the number of such vectors is equal to

$$\lambda_\rho \ = \ \binom{\rho - e - 1}{e + 1} \cdot \lambda_d.$$

Indeed, for any choice of $e + 1$ positions from $\rho - e - 1$, there are exactly λ_d different vectors $\mathbf{c}_i$ from C_d, with these fixed $e + 1$ positions on $\mathrm{supp}(\mathbf{y})$ and the rest $e + 1$ positions on $J_d^{(2)}$. This gives the expression (24) for λ_ρ.

By the arguments above the number $\xi_\rho(i)$ does not depend on $i \in \mathrm{supp}(\mathbf{y})$ for chosen $\mathbf{y}$. To find this number, we fix one position on $\mathrm{supp}(\mathbf{y})$ and choose the other e positions from the rest $\rho - e - 2$ positions by all possible ways:

$$\xi_\rho(i) \ = \ \binom{\rho - e - 2}{e} \cdot \lambda_d.$$

It is clear, that we will have the same expression for $\xi_\rho(i)$, if we choose as $\mathbf{v}^*$ any other vector $\mathbf{v}$ from $C(\rho)_\rho$, covering $\mathbf{x}$. Thus, $\xi_\rho(i)$ is the same (i.e. $\xi_\rho(i) = \xi_\rho$) for all i from $J_\rho^{(1)}$. Counting by two different ways the all number of nonzero positions of vectors $\mathbf{v} \in C(\rho)_\rho$, covering $\mathbf{x}$, we obtain that $\xi_\rho \ell_\rho = (\rho - e - 1)\lambda_\rho$, which gives (25). $\square$

Lemma 15 *We have that*

$$\ell_\rho = \frac{(\rho - e - 1)^2}{e+1} \tag{26}$$

and

$$\ell_\rho \geq \ell_d + \rho - e - 1. \tag{27}$$

PROOF. Returning to the proof of the previous lemma, we have that

$$\ell_\rho = \frac{\lambda_\rho}{\xi_\rho} \cdot (\rho - e - 1).$$

Now the expression for ℓ_ρ follows, if we take into account expressions for λ_ρ and ξ_ρ from the lemma above.

To prove the inequality we deduce from (23) that $\ell_\rho = |J_\rho^{(1)}| = |J_d^{(2)}| + \rho - e - 1$. The bound follows now from simple observation that $|J_d^{(2)}| \geq |J_d^{(1)}| = \ell_d$ (indeed, in partition $P(\mathbf{v}^*, \mathbf{y})$ for any $e+1$ fixed nonzero positions of $\mathbf{y}$, we should have exactly λ_d disjoint vectors of weight $e+1$ on $J_d^{(2)}$). $\square$

Now we have to consider the cases $\rho = 2e+2$ and $\rho \geq 2e+3$ separately. We start from the case $\rho \geq 2e+3$. In partition $P(\mathbf{v}^*, \mathbf{y})$ let $\mathbf{z}_i \in W(\mathbf{y}^*)$, $i = 1, 2$. Denote by ξ_1 (respectively, by ξ_2) the number of words from C which are at distance $\rho - e - 2$ from $\mathbf{z}_1$ (respectively, from $\mathbf{z}_2$), where $\mathbf{z}_1$ is covered by $\mathbf{y}$ (respectively, $\mathbf{z}_2$ has one nonzero position on $J_d^{(2)}$).

Lemma 16 *Let* $\rho \geq 2e+3$. *Then*

$$\xi_1 \geq \lambda_d \cdot \binom{\rho - e - 2}{e+1} + 1 \tag{28}$$

and

$$\xi_2 \geq \lambda_\rho \cdot \frac{e+1}{\ell_\rho - \rho + e + 1}. \tag{29}$$

PROOF. Since $\mathrm{wt}(\mathbf{z}_1) = \rho - e - 2$, it is at distance $\rho - e - 2$ from zero codeword. Now, for any choice of $e+1$ positions in $\mathrm{supp}(\mathbf{z}_1)$, there are exactly λ_d codewords from C_d at the distance $\rho - e - 2$. There might be also some codewords from C_{d+2} at the same distance from $\mathbf{z}_1$, which we can not evaluate. Hence, we conclude that ξ_1 is not less than the expression (28) of the lemma.

Similarly, for the number ξ_2, for any $\rho - e - 1$ nonzero positions of $\mathbf{y}$, there are exactly λ_d codewords from C_d where each has exactly $e+1$ nonzero positions on $J_d^{(2)}$. We can lower bound the number ξ_2 taking average contribution of these codewords from C_d to one position of $J_d^{(2)}$. This gives the following lower bound (again we do not know the number of possible codewords from C_{d+2}):

$$\xi_2 \;\geq\; \lambda_d \cdot \binom{\rho - e - 1}{e+1} \cdot \frac{e+1}{|J_d^{(2)}|}.$$

But by (23)

$$|J_d^{(2)}| + \rho - e - 1 \;=\; |J_\rho^{(1)}|.$$

Recalling that $|J_\rho^{(1)}| = \ell_\rho$, we obtain from these two expressions above the second inequality of the lemma. $\square$

Now we return to the proof of Theorem 13 for the case $\rho \geq 2e+3$. Consider the partition $P(\mathbf{x})$. Let $\mathbf{z}_3 \in W(\mathbf{x})$ contain one nonzero position on $J_\rho^{(1)}$. Then we know that there are exactly ξ_ρ vectors from $C(\rho)_\rho$ at distance $\rho - e - 2$ from $\mathbf{z}_3$, and there are no any vectors from $C(\rho)$ at this distance (see Lemma 14). But C is completely regular code and, since $C(\rho)$ is a shift of C (Theorem 10), all these numbers ξ_ρ, ξ_1, and ξ_2 should be equal. This implies the following inequality:

$$\max\left(\lambda_d \cdot \binom{\rho - e - 2}{e+1} + 1,\;\; \frac{(e+1)\lambda_\rho}{\ell_\rho - \rho + e + 1}\right) \;\leq\; \xi_\rho. \tag{30}$$

Consider the first inequality

$$\lambda_d \binom{\rho - e - 2}{e+1} + 1 \;\leq\; \xi_\rho.$$

which is equivalent to the following one:

$$\lambda_d \binom{\rho - e - 2}{e + 1} \; < \; \xi_\rho .$$

Taking into account (25), the last inequality is reduced to the following one:

$$\ell_\rho \; < \; \frac{(\rho - e - 1)^2}{\rho - 2e - 2}. \tag{31}$$

The second inequality

$$\frac{(e+1)\lambda_\rho}{\ell_\rho - \rho + e + 1} \; \leq \; \xi_\rho$$

implies that

$$\ell_\rho \; \geq \; \frac{(\rho - e - 1)^2}{\rho - 2e - 2}. \tag{32}$$

Comparing (31) and (32), we obtain a contradiction. We conclude that for the case $\rho \geq 2e + 3$ there is no such code C, which satisfies to the conditions of the theorem.

Now we continue the proof of theorem for the case $\rho = 2e + 2$. Consider the intersection numbers a_i, b_i, c_i of C for $i = \rho/2 = e + 1$. As we mentioned already, by Lemma 7 the intersection numbers $a_{\rho-i}, b_{\rho-i}, c_{(\rho-i}$ are reversed of a_i, b_i, c_i, i.e.

$$b_i \; = \; c_{\rho-i} \;\; \text{and} \;\; c_i \; = \; b_{\rho-i}.$$

For the case $\rho = 2e + 2$ and $i = e + 1$ all these numbers should be equal, since $C(\rho) = C + \mathbf{1}$ by Theorem 10, i.e. we should have

$$b_{e+1} \; = \; c_{e+1}.$$

Using (19), we deduce that $\ell_\rho = \ell_d + e + 1$. But for the case $\rho = 2e + 2$, the expression (26) gives $\ell_\rho = e + 1$, i.e. we obtain that $\ell_d = 0$, which is impossible, since $J_d^{(1)}$ is nonempty. Thus, we obtain a contradiction. Therefore, such code C with $\rho = 2e + 2$ can not exist for any $e \geq 1$. This means, that if such code exists it should have $\rho \leq 2e + 1$.

Now combining this with Theorem 10, we conclude that $\rho = 2e + 1$. Hence $C(\rho)_\rho$ is a $(e+1)$-design $T(n, 2e+1, e+1, \lambda_\rho)$ and a constant weight code with minimum distance $2e + 2$, which is possible if and only if $\lambda_\rho = 1$. Thus, $C(\rho)_\rho$ is a Steiner system $S(n, 2e+1, e+1)$. The theorem is proved. □

4 On non-antipodal completely regular codes and binary perfect codes

The next two statements give a characterization of all nontrivial binary non-antipodal completely regular codes with odd or even minimum distance d.

Theorem 17 *Let C be a nontrivial (i.e. $|C| > 2$) completely regular code with parameters n, $d = 2e + 1 \geq 3$, and ρ. If $\mathbf{0} \in C$ and $\mathbf{1} \notin C$, then C is a punctured half of perfect code C' and*

$$C' = C^* \cup C^*(\rho),$$

where C^* is obtained from C by extension with even parity checking, $C^*(\rho)$ is the covering set of C^*, and C' is a binary perfect code with parameters $n' = n + 1$, $d' = 2e + 1$ and $\rho' = e$.

PROOF. Let C^* obtained from C by even parity checking, i.e. it is a code with $d = 2e + 2$. From Theorem 12 we have that $\rho = 2e$. Denote by $C^*(\rho)$ the covering set of C^*, obtained from $C(\rho)$ by odd parity checking. Then C^* has covering radius $\rho^* = \rho + 1 = 2e + 1$. It is easy to see also that $C^*(\rho)$ is a shift of C^* by $\mathbf{1}$. Define a new code C' as a union of C^* and $C^*(\rho)$. By definition of covering set the code C' has minimum distance $d' = \rho^* = 2e + 1$. Now we have to show only that this new code has the covering radius $\rho' = e$. The lower bound $\rho' \geq e$ is trivial. To see that $\rho' \leq e$, recall the proof of Theorem 12. As we proved there the union $C^*(\rho)_{\rho+1} \cup C^*(\rho)_\rho$ form the Steiner system $S(n+1, 2e+1, e+1)$. This means that any vector $\mathbf{z}$ of weight $e+1$ is covered by exactly one vector from $S(n+1, 2e+1, e+1)$, or by $C^*(\rho)_{\rho+1}$. This implies that $\rho' \le e$. Thus $\rho' = e$. $\square$

Theorem 18 *Let C be a nontrivial (i.e. $|C| > 2$) completely regular code with parameters n, $d = 2e+2 \ge 4$, and ρ. If $\mathbf{0} \in C$ and $\mathbf{1} \notin C$, then C is a half of perfect code, and C',*

$$C' \;=\; C \cup C(\rho),$$

i.e. a union of C and its covering set $C(\rho)$, is a binary perfect code with parameters $n' = n$, $d' = 2e+1$, and $\rho' = e$ where $e = \lfloor (d-1)/2 \rfloor$.

PROOF. By Theorem 13 we have that $\rho = 2e+1$. Define a new code C' (with minimum distance d' and covering radius ρ'), taking a union of C and $C(\rho)$. Since $C(\rho)$ is a shift of C, we deduce that $d' = \rho = 2e+1$.

We claim that C' is a perfect code. To have it we have to show that $\rho' = e$. First, it is clear that $\rho' \ge e$ (indeed, $\rho = 2e+1$ and C and $C(\rho)$ are codes with minimum distance $2e+2$). To see that $\rho' \le e$, recall the proof of Theorem 13. In terms of partition $P(\mathbf{x})$, induced by any vector $\mathbf{x}$ of weight $e+1$, the inequality $\rho' \le e$ is the same as existence of some $\mathbf{v} \in C(\rho)_\rho$ covering $\mathbf{x}$. But this follows from the fact that $C(\rho)_\rho$ is a $(e+1)$-design. Thus, C' is a perfect code, and C is a half of a perfect code. $\square$

The following example shows that for trivial completely regular codes with $|C| = 1$ this last theorem is not valid.

Example 19 *Consider a trivial code C, consisting of one vector in $\mathbb{F}^n$, which is completely regular non-antipodal code with $\rho = n$. Let $C = \{(0,0,...,0)\}$ for n multiple of 4. The intersection array of L looks as follows:*

$$a_i = 0,\ b_i = n - i,\ c_i = i,\ \ i = 0, 1, ..., n.$$

By Theorem 10 above the set $C(\rho)$ *is the complementary vector* $\mathbf{1} = (1, 1, ..., 1)$*. The middle row for* $i = n/2$ *is symmetric* $b_{n/2} = c_{n/2} = n/2$*. as it should be, since to this code we can add the complementary vector* $(1, 1, ..., 1)$ *and to obtain a completely regular code again with two codewords and with even covering radius* $\rho' = n/2$ *(but not perfect code with odd covering radius, how we have in Theorem 17).*

Now we have the following natural question: *which half of a perfect code* C' *is a code* C*?* Since $\mathbf{0}$ does belong to C, it is quite natural to suggest that it is an even subcode of C'. The next statement answer this question for known binary perfect codes, i.e. for codes with Hamming parameters and for the binary Golay code (since these are the only nontrivial binary perfect codes [15], [17]).

Theorem 20 *Let* C *be a nontrivial (i.e.* $|C| > 2$*) completely regular binary code with parameters* n*,* $d = 2e + 2 \geq 4$*, and* ρ*. Assume that* $\mathbf{0} \in C$*, and* $\mathbf{1} \notin C$*. Then* $\rho = 2e + 1$ *and* C *is the even half part of a perfect code* C' *with minimum distance* $d(C') = 2e + 1$*.*

PROOF. By Theorem 17 code C is a half of a e-perfect code C' and the minimum distance of C is $d = 2e+2$. First consider 1-perfect codes (i.e. codes with $d = 3$. Let $(\mu_0, \mu_1, \ldots, \mu_n)$ be the weight distribution of 1-perfect code C' with zero codeword. It is well known that $\mu_i \neq 0$ for all region from 0 to n, except $i = 1, 2, n-1, n-2$. The following two properties follow from the definition of a perfect binary code. For any neighbor sets C'_i and C'_{i+1} where $i = 3, 4, \ldots, n-4$:

$(Q.1)$ *for any* $\mathbf{c} \in C'_i$ *there are codewords from* C'_{i+1} *at distance* 3 *from* $\mathbf{c}$*;*

$(Q.2)$ *for any* $\mathbf{c} \in C'_{i+1}$ *there are codewords from* C'_i *at distance* 3 *from* $\mathbf{c}$*.*

It is clear, that the even half of C' is the code C with cardinality $|C| = |C'|/2$ and with minimum distance 4, as well as, the rest part $C'' = C' \setminus C$, which is a shift of C. Now we want to prove that it is the only possibility. Since $\mathbf{0} \in C$, we deduce that C can not contain any word from C'_3. Hence we choose for C all words from C'_4. If not, the words which are not chosen will have distance 3 from C' (property $(Q.2)$). But now, since C contains all words from C'_4, we can not choose any word from C'_5 (property $(Q.2)$). Continuing in this way we obtain that C contains of all codewords of C' of even weight.

For the Golay $[23, 12, 7]$-code C' the proof is similar. □

Thus, after Theorems 12, 13 and 20, any nontrivial non-antipodal completely regular code with $d \geq 3$ is a half of a perfect code, or is a punctured half of it. But the only nontrivial binary perfect codes are the binary Golay $[23, 12, 7]$ code and $(n = 2^m - 1, N = 2^{n-m}, 3)$ codes with parameters of Hamming codes [15], [17]. We have, therefore, from the results above the following result.

Theorem 21 *Let C be a nontrivial (i.e. $|C| > 2$) non-antipodal completely regular binary code with parameters n, $d \geq 3$ and ρ. Then there are exactly four cases:*
If $d = 2e + 2$, then:
1). *C is a half of binary perfect Golay code and $n = 23$, $d = 8$ and $\rho = 7$.*
2). *C is a half of binary perfect code with Hamming parameters, i. e. $n = 2^m - 1$, $d = 4$, $\rho = 3$, where $m = 3, 4, \ldots$.*
If $d = 2e + 1$, then:
3). *C is a punctured half of binary perfect Golay code and $n = 22$, $d = 7$ and $\rho = 6$.*
4). *C is a punctured half of binary perfect code with Hamming parameters, i. e. $n = 2^m - 2$, $d = 3$, $\rho = 2$, where $m = 3, 4, \ldots$.*

PROOF. Let C be a non-antipodal completely regular code with minimum distance d. If $d = 2e + 1$, then by Theorem 12 we deduce that $\rho = 2e$, and by Theorem 17, we obtain that C^* is a punctured half of a perfect code. In particular, by Theorem 20, we see that the code C^* (obtained by extension of C) with zero word is the even part of perfect code. This means that original code C is either the shortened half of the Golay $[23, 12, 7]$ code, i.e. the case 3), or the punctured halves of the perfect codes with Hamming parameters, i.e. the case 4), found in [14].

If C has $d = 2e+2$, by Theorem 13 we deduce that $\rho = 2e+1$, and by Theorem 18, we obtain that C^* is a half of perfect code. In this case we obtain, either a half of Golay code, i.e. the case 1), or halves of binary perfect codes with Hamming parameters, i.e. the case 2). □

Now taking halves and punctured halves of known binary perfect codes, we obtain new completely regular, completely transitive, and uniformly packed in the sense of [1].

Corollary 22 *[4] Let G' be a binary perfect Golay $[23, 12, 7]$-code. Denote by G its subcode, formed by all codewords of even (respectively, odd) weight. Then:*
(i) G is a completely regular $(23, 2^{11}, 8)$-code with $\rho = 7$ and intersection array

$$(23, 22, 21, 20, 3, 2, 1; 1, 2, 3, 20, 21, 22, 23).$$

(ii) G is a completely transitive code.
(iii) G is a uniformly packed code in the sense of [1] with parameters α_i, $i = 0, 1, ..., 7$:

$$\alpha_0 = 1, \quad \alpha_1 = \frac{7}{23}, \quad \alpha_2 = \frac{3}{11}, \quad \alpha_3 = \frac{179}{7{\cdot}11{\cdot}23},$$

$$\alpha_4 = \frac{29}{5{\cdot}7{\cdot}11}, \quad \alpha_5 = \frac{47}{7{\cdot}11{\cdot}23}, \quad \alpha_6 = \frac{1}{7{\cdot}11}, \quad \alpha_7 = \frac{1}{11{\cdot}23}.$$

Corollary 23 *[4] Let G' be a binary perfect Golay $[23, 12, 7]$-code. Denote by*

G its subcode, formed by all codewords of even (respectively, odd) weight and by $G^{(1)}$ the code, obtained by puncturing one position of G. Then:

(i) $G^{(1)}$ is a completely regular $(22, 2^{11}, 7)$-code with $\rho = 6$ and intersection array

$$(22, 21, 20, 3, 2, 1; 1, 2, 3, 20, 21, 22).$$

(ii) $G^{(1)}$ is a completely transitive code.

(iii) $G^{(1)}$ is a uniformly packed code in the sense of [1] with parameters α_i, $i = 0, 1, ..., 6$:

$$\alpha_0 = 1, \qquad \alpha_1 = \alpha_2 = \tfrac{3}{11},$$

$$\alpha_3 = \alpha_4 = \tfrac{29}{5\cdot 7\cdot 11}, \;\; \alpha_5 = \alpha_6 = \tfrac{1}{7\cdot 11}.$$

We remark that the code $G^{(1)}$ is also uniformly packed of order 3 in the sense of [8].

Corollary 24 *[4] Let H' be a binary 1-perfect code of length $n = 2^m - 1 \geq 7$. Denote by H its even (respectively, odd) subcode. Then:*

(i) H is completely regular with covering radius $\rho = 3$ and with intersection array $(n, n-1, 1; 1, n-1, n)$.

(ii) H is uniformly packed in the wide sense, i.e. in the sense of [1] with parameters α_i:

$$\alpha_0 = 1, \;\; \alpha_1 = \frac{3}{n}, \;\; \alpha_2 = \frac{2}{n-1}, \;\; \alpha_3 = \frac{6}{n(n-1)}.$$

(iii) If H' is completely transitive, then H is completely transitive also.

Corollary 25 *[14] Let H' be a binary 1-perfect code of length $n = 2^m - 1 \geq 7$. Denote by H its even (respectively, odd) subcode and by $H^{(1)}$ the code, obtained by puncturing one position of H. Then:*

(i) $H^{(1)}$ is completely regular with covering radius $\rho = 2$ and with intersection array $(n-1, 1; 1, n-1)$.

(ii) $H^{(1)}$ *is uniformly packed in the narrow sense, i.e. in the sense of [14] with parameters* $\alpha_0 = 1$ *and* $\alpha_1 = \alpha_2 = 2/(n-1)$.

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