New completely regular and completely transitive binary codes

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***Abstract*— Non-antipodal binary completely regular codes are considered. The only such codes come from binary perfect codes and specifically such new codes with covering radius $\rho = 3$ and $\rho = 7$ are constructed. In particular, a new binary completely regular code with minimal distance $d = 8$ and covering radius $\rho = 7$ has been built, which disproves the known conjecture of Neumaier from 1992, that the extended binary Golay $[24, 12, 8]$-code is the only binary completely regular code with $d \geq 8$ and, also, an infinite family of binary completely regular codes with $d = 4$ and $\rho = 3$ is founded. It is proved that some of these new codes are also new completely transitive codes and, of course, new uniformly packed codes in the wide sense. As a corollary of the result on nonexistence of nontrivial binary perfect codes it is obtained that there are no unknown nontrivial non-antipodal completely regular binary codes with minimal distance $d \geq 3$.**

***Index Terms*— non-antipodal codes, completely regular codes, completely transitive codes, perfect codes, additive codes, Z_4-linear codes.**

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I. INTRODUCTION

Let $\mathbb{F}^n$ be the n-dimensional vector space over $\mathbb{F} = GF(2)$. The *Hamming weight*, $\mathrm{wt}(\mathbf{v})$, of a vector $\mathbf{v} \in \mathbb{F}^n$ is the number of its nonzero coordinates. The *Hamming distance* between two vectors $\mathbf{v}, \mathbf{u} \in \mathbb{F}^n$ is $d(\mathbf{v}, \mathbf{u}) = \mathrm{wt}(\mathbf{v} + \mathbf{u})$. A *(binary)* (n, N, d)*-code* C is a subset of $\mathbb{F}^n$ where n is the *length*, d is the *minimal distance*, and $N = |C|$ is the *cardinality* of C. Given any vector $\mathbf{v} \in \mathbb{F}^n$, its *distance to the code* C is

$$d(\mathbf{v}, C) = \min_{\mathbf{x} \in C}\{d(\mathbf{v}, \mathbf{x})\}$$

and the *covering radius* of the code C is

$$\rho = \max_{\mathbf{v} \in \mathbb{F}^n}\{d(\mathbf{v}, C)\}$$

Given two sets $X, Y \subset \mathbb{F}^n$, define: their sum $X + Y = \{\mathbf{x} + \mathbf{y} : \ \mathbf{x} \in X, \ \mathbf{y} \in Y\}$ and their minimum distance $d(X, Y) = \min\{d(\mathbf{x}, \mathbf{y}) : \ \mathbf{x} \in X, \ \mathbf{y} \in Y\}$.

We write $X + \mathbf{x}$ instead of $X + \{\mathbf{x}\}$. For a given vector $\mathbf{x} \in \mathbb{F}^n$ let $\bar{\mathbf{x}}$ be the complementary vector, i.e. $d(\mathbf{x}, \bar{\mathbf{x}}) = n$. For a given set $X \subset \mathbb{F}^n$ define the complementary set $\bar{X} = \{\bar{\mathbf{x}} : \ \mathbf{x} \in X\}$.

For a given code C with covering radius $\rho = \rho(C)$ define

$$C(i) \ = \ \{\mathbf{x} \in \mathbb{F}^n : \ d(\mathbf{x}, C) = i\}, \ \ i = 1, 2, ..., \rho.$$

Recently, starting from the celebrated article [11] about $\mathbb{Z}_4$-linear codes has spawned significant research relating such codes to classical well-known codes which were not linear, such as the Preparata codes, Kerdock codes, etc. The tool used to view a $\mathbb{Z}_4$ code as a binary code is the Gray map φ, which takes a $\mathbb{Z}_4$ symbol and maps it into a pair of bits as follows: $\varphi(0) = 00$, $\varphi(1) = 01$, $\varphi(2) = 11$, $\varphi(3) = 10$. The Gray map is an isometry which transforms Lee distances defined in the quaternary codes to Hamming distances defined in the binary ones.

More general than the $\mathbb{Z}_4$-linear codes are the additive codes (see [4], [8]), which, roughly speaking, can be seen as codes with some coordinates in $\mathbb{Z}_2$ and other coordinates in $\mathbb{Z}_4$ in such a way that the code is an Abelian subgroup of $(\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta, +)$, where $+$ means the usual additive operation on $\mathbb{Z}_2$ and $\mathbb{Z}_4$.

We assume that the binary code C always contains the zero vector, unless stated otherwise. Let $D = C + \mathbf{x}$ be a *shift* of C. The *weight* $\mathrm{wt}(D)$ of D is the minimum weight of the codewords of D. For an arbitrary shift D of weight $i = \mathrm{wt}(D)$ denote by $\mu(D) = (\mu_0(D), \mu_1(D), ..., \mu_n(D))$ its weight distribution, where $\mu_i(D)$ denotes the number of words of D of weight i. So $\mu(C) = (\mu_0(C), ..., \mu_n(C))$ is the weight distribution of C. If $\mu(C)$ is the same for any choice of zero word of C, then C is *distance invariant*. Denote by C_j (respectively, D_j, and $C(i)_j$) the subset of C (respectively, of D and $C(i)$), formed by all words of the weight j. In our terminology $\mu_i(C) = |C_i|$ and $\mu_i(D) = |D_i|$. If C is linear (or additive), we say that $D = C + \mathbf{x}$ is a *coset* of C by $\mathbf{x}$.

Definition 1: A binary code C with covering radius ρ is called *completely regular* if the weight distribution of any shift D of weight i, $i = 0, 1, ..., \rho$ of C is uniquely defined by the minimum weight of D, i.e. by the number $i = \mathrm{wt}(D)$.

We use also the definition of completely regularity given in [15]. A code C is a completely regular code if, for all $l \geq 0$, every vector $x \in C(l)$ has the same number c_l of neighbors in $C(l-1)$ and the same number b_l of neighbors in $C(l+1)$. Also, define $a_l = n - b_l - c_l$ and note that $c_0 = b_\rho = 0$. Define by $\{b_0, \ldots, b_{\rho-1}; c_1, \ldots, c_\rho\}$ the intersection array of C and by L the intersection matrix of C:

$$L = \begin{pmatrix} a_0 & b_0 & 0 & 0 & \cdots & 0 \\ c_1 & a_1 & b_1 & \cdots & 0 & 0 \\ 0 & c_2 & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \vdots & \ddots & \ddots & b_{\rho-1} \\ 0 & 0 & \cdots & \cdots & c_\rho & a_\rho \end{pmatrix}.$$

For a binary code C let $\mathrm{Perm}(C)$ be its permutation stabilizer group. For any $\theta \in \mathrm{Perm}(C)$ and any coset $D = C + \mathbf{x}$ of C define the action of θ to D as: $\theta(D) = C + \theta(\mathbf{x})$.

Definition 2: Let C be a binary additive code with covering radius ρ. The code C is called *completely transitive*, if the set $\{C + \mathbf{x} : \mathbf{x} \in \mathbb{F}^n\}$ of all different cosets of C is partitioned under action of $\mathrm{Perm}(C)$ exactly into $\rho + 1$ orbits.

Since two cosets in the same orbit should have the same weight distribution, it is clear, that any completely transitive code is completely regular.

Let C be a binary e-error-correcting code. It has been conjectured for a long time that if C is a completely regular code and $|C| > 2$, then $e \leq 3$. In fact, this conjecture has been also stated for non binary codes. Moreover, in [15] it is conjectured that the only completely regular code C with $|C| > 2$ and $d \geq 8$ is the well known extended binary Golay $[24, 12, 8]$-code with $\rho = 4$. As we know from the results [19] and [21] for the case $\rho = e$ and [20] (see also [18], [10]) for the case $\rho = e + 1$, any such nontrivial unknown code should have a covering radius $\rho \geq e+2$. For the special case of completely regular codes, for linear completely transitive codes [16], the problem of existence is solved: in our previous papers [2] and [3] we proved that for $e \geq 4$ such nontrivial codes do not exist.

In the separate paper [6] we give a complete characterization of binary nontrivial non-antipodal completely regular codes with minimal distance $d \geq 3$. The only such codes are formed by halves of binary perfect codes. As a corollary of the result on nonexistence of nontrivial binary perfect codes we have obtained that there are no unknown nontrivial non-antipodal completely regular binary codes with minimal distance $d \geq 3$.

Using these results we obtain here, in this paper, that there exist new binary completely regular codes. In particular, we prove that the $(23, 2^{11}, 8)$ code, which is a half of the perfect Golay $[23, 12, 7]$ code, is a completely regular code with minimal distance $d = 8$, with intersection array

$$(23, 22, 21, 20, 3, 2, 1; 1, 2, 3, 20, 21, 22, 23)$$

and covering radius $\rho = 7$. This result implies that the conjecture of Neumaier [15] (that the binary Golay $[24, 12, 8]$-code is the only binary completely regular code with $d \geq 8$) is not valid. This new completely regular code is also completely transitive. Halves of binary perfect $(n = 2^m - 1, N = 2^{n-m}, 3)$ codes with Hamming parameters also give a new infinite family of completely regular codes with $\rho = 3$ and intersection array $(n, n-1, 1; 1, n-1, n)$. All these new codes are uniformly packed in the wide sense, i.e. in the sense of [1]. Many of these new completely regular codes, obtained from binary additive perfect codes are new linear and nonlinear ($\mathbb{Z}_4$-linear or additive) completely transitive codes. Using recent results on enumeration of $\mathbb{Z}_4$-linear [12] and additive [4] binary perfect codes with $d = 3$, we obtain an infinite family of nonequivalent completely regular and completely transitive codes. The same results are valid for q-ary perfect codes, under certain conditions on original codes. In particular, from ternary Golay code we obtain new ternary completely regular code with minimal distance 6, with covering radius 5 and with intersection array $(22, 20, 18, 2, 1; 1, 2, 9, 20, 22)$. This code is also completely transitive and uniformly packed in the wide sense [1]. These results will be published in the separate paper [5].

The paper is organized as follows. In Section 2 we give some preliminary results concerning completely regular and completely transitive codes. Section 3 is dedicated to non-antipodal completely regular codes. The only such codes are formed by even (or odd) codewords of binary perfect codes. Among other results this gives a very natural explanation why for a binary nontrivial perfect code of length n and with covering radius ρ both numbers n and ρ are odd. In Section 4 we give new binary completely regular, uniformly packed in the wide sense [1], and completely transitive codes, obtained from binary perfect codes.

II. Preliminary results

We give some definitions, notations and results which we will need.

Definition 3: Let C be any binary code of length n and let ρ be its covering radius. We say that such a code is *uniformly packed* in the wide sense, i.e. in the sense of [1], if there exist rational numbers $\alpha_0, \ldots, \alpha_\rho$ such that for any $\mathbf{v} \in F^n$

$$\sum_{k=0}^{\rho} \alpha_k \, f_k(\mathbf{v}) = 1 \,, \qquad (1)$$

where $f_k(\mathbf{v})$ is the number of codewords at distance k from $\mathbf{v}$. We say that such a code is *strongly uniformly* *packed*, or uniformly packed in the sense of [18], if $\rho = e+1$ and $\alpha_e = \alpha_{e+1}$, where $e = \lfloor (d-1)/2 \rfloor$.

The *support* of $\mathbf{v} \in F^n$, $\mathbf{v} = (v_1, \ldots, v_n)$ is $supp(\mathbf{v}) = \{ \ \ell \mid v_\ell \neq 0 \ \}$. Say that a vector $\mathbf{v}$ *covers* a vector $\mathbf{z}$ if $supp(\mathbf{z}) \subseteq supp(\mathbf{v})$.

Definition 4: Recall that a *t-design* $T(n, w, t, \beta)$ is a set of binary vectors of length n and weight w such that for any binary vector $\mathbf{z}$ of weight t, $1 \leq t \leq w$, there are precisely β vectors $\mathbf{v}_i, \ i = 1, ..., \beta$, of $T(n, w, t, \beta)$ each of them covering the vector $\mathbf{z}$. For the case $\beta = 1$ the design $T(n, w, t, 1)$ is a Steiner system, which we denote by $S(n, w, t)$.

Definition 5: We say that a binary code C is *even* (respectively, *odd*) if all its codewords have even (respectively, *odd*) weights.

Definition 6: We say that a binary code C with minimum distance d and with zero codeword has t-design property, if any nonempty set C_j, $d \leq j \leq n$, is a t-design.

The following well known fact directly follows from the definition of completely regular code.

Lemma 1: Let C be a completely regular code with minimum distance d and with zero codeword. Then C has t-design property, where $t = e$, if $d = 2e+1$ and $t = e+1$, if $d = 2e+2$.

The next statement can be found in [15].

Lemma 2: If C is completely regular with covering radius ρ, then $C(\rho)$ is also completely regular, with reversed intersection array.

III. On non-antipodal completely regular codes and binary perfect codes

Definition 7: The code C is called *antipodal*, if for any $\mathbf{c} \in C$ the complementary vector $\bar{\mathbf{c}} = \mathbf{c} + \mathbf{1}$ is also a codeword of C.

It is clear that a distance invariant code C, containing the zero vector $\mathbf{0}$, is antipodal, if it contains the all-one vector $\mathbf{1}$.

The first natural question is: *does any completely regular code contain the vector* $\mathbf{1}$? We start from the statement, which defines the place of the all one vector $\mathbf{1}$ for arbitrary binary code.

Lemma 3: Let C be any binary code. Then C and $C(\rho)$ are antipodal or not simultaneously.

Proof. Let C be any binary code, and let $C(\rho)$ be the corresponding covering set of C. Assume that C is antipodal. To see $C(\rho)$ is antipodal we take $\mathbf{v} \in C(\rho)$ and prove that $\mathbf{1} + \mathbf{v} \in C(\rho)$. In order to do this we observe that $d(\mathbf{1} + \mathbf{v}, C) = \rho$, since

$$d(\mathbf{1} + \mathbf{v}, C) = d(\mathbf{v}, \mathbf{1} + C) = d(\mathbf{v}, C) = \rho.$$

The statement follows now since the antipodality of $C(\rho)$ implies the antipodality of C by reversing of C and $C(\rho)$. $\triangle$

Now we formulate one lemma and one theorem, which give a complete characterization of binary non-trivial non-antipodal completely regular codes. The proofs of these results is a subject of separate paper [6].

Lemma 4: [6] Let C be a completely regular code with covering radius ρ, with minimal distance $d \geq 3$ and with zero vector. If $\mathbf{0} \in C$ and $\mathbf{1} \notin C$, then $\rho \geq 2e$, where $e = \lfloor (d-1)/2 \rfloor$, the vector $\mathbf{1}$ belongs to $C(\rho)$ and $C + \mathbf{1} = C(\rho)$.

Lemma 5: Let C and its (even or odd) extension C^* be completely regular codes of lengths n and $n+1$ and with covering radii ρ and $\rho+1$, respectively. Then C and C^* are antipodal or not simultaneously.

Proof. Assume C is antipodal, but C^* is not, i.e. $\mathbf{0} \in C$ and $\mathbf{1} \in C$. Then by Lemma 4 we have that $\mathbf{1} \in C^*(\rho+1)$, and, therefore, $\mathbf{1} \in C(\rho)$, i.e. a contradiction. Hence C^* is antipodal too. If C^* is antipodal, then clearly C is antipodal too. $\triangle$

The next theorem gives a characterization of all non-trivial binary non-antipodal completely regular codes.

Theorem 1: [6] Let C be a nontrivial (i.e. $|C| > 2$) completely regular code with parameters n, $d \geq 3$, and ρ. If $\mathbf{0} \in C$ and $\mathbf{1} \notin C$, then we have that $d = 2e+2$, where $e = \lfloor (d-1)/2 \rfloor$, ρ is odd, C is a half of perfect code, and $C' = C \cup C(\rho)$, i.e. a union of C and its covering set $C(\rho)$, is a binary perfect code with parameters $n' = n$, $d' = 2e+1$, and $\rho' = e$.

The following example shows that for trivial completely regular codes with $|C| = 1$ this theorem is not valid.

Example: Consider a trivial code C, consisting of one vector in $\mathbb{F}^n$, which is completely regular non-antipodal code with $\rho = n$. Let $C = \{(0000)\}$ for $n = 4$. The intersection matrix L looks as follows:

$$L = \begin{pmatrix} 0 & 4 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 4 & 0 \end{pmatrix}.$$

By Lemma 4 above the set $C(\rho)$ is the complementary vector $\mathbf{1} = (1111)$. As we see $b_2 = c_2$ as it should be, since to this code we can add the complementary vector (111111) and to obtain a completely regular code again with two codewords and with even covering radius $\rho' = 2$ (but not perfect code with odd covering radius, how we have in Theorem 1). ∎

Now we have the following natural question: *which half of a perfect code C' is a code C?* Since $\mathbf{0}$ does belong to C, it is quite natural to suggest that it is an even subcode of C'. The next statement answer this question.

Theorem 2: Let C be a nontrivial (i.e. $|C| > 2$) completely regular binary code with parameters n, $d \geq 3$, and ρ. Assume that $\mathbf{0} \in C$, $\mathbf{1} \notin C$. Then C is the even half part of a perfect code with $d = d(C) = 2e+2$ and $\rho = 2e+1$ where $e = \lfloor (d-1)/2 \rfloor$.

Proof. By Theorem 1 code C is a half of a e-perfect code C' and the minimum distance of C is $d = 2e+2$. First consider 1-perfect codes. Let $(\mu_0, \mu_1, \ldots, \mu_n)$ be the weight distribution of 1-perfect code C' with zero codeword. It is well known that $\mu_i \neq 0$ for all value of i from 0 to n, except four zero values, namely, $\mu_i = \mu_{n-i} = 0$ for $i = 1, 2$. The following two properties follow from the definition of a perfect binary code. For any two neighboring sets C'_i and C'_{i+1} where $i = 3, 4, \ldots, n-4$:

$(Q.1)$ *for any* $\mathbf{c} \in C'_i$ *there are codewords from* C'_{i+1} *which are at distance* 3 *from* $\mathbf{c}$*;*

$(Q.2)$ *for any* $\mathbf{c} \in C'_{i+1}$ *there are codewords from* C'_i *at distance* 3 *from* $\mathbf{c}$.

It is clear, that taking all codewords of even weight from C' we obtain a subcode C with cardinality $|C| = |C'|/2$ and with minimum distance 4, as well as the rest part $C'' = C' \setminus C$, which is a shift of C, is a subcode of C' with minimal distance four too. Now we want to prove that it is the only possibility, i.e. the only subcode of C' of half cardinality, with zero codeword, with minimal distance four (and such that $C'' = C' \setminus C$ has also minimal distance four) is the even subcode of C'. Since $\mathbf{0} \in C$, we deduce that C can not contain any word from C'_3. Now we have to choose for C all words from C'_4. If not, the words which are not chosen will have minimal distance three in C'' (property $(Q.2)$). But now, since C contains all words from C'_4, we can not choose any word from C'_5 (property $(Q.2)$). Continuing in this way we obtain that C contains of all codewords of C' of even weight.

For the Golay $[23, 12, 7]$-code C' the proof is similar. $\triangle$

Thus, any nontrivial, non-antipodal completely regular code with $d \geq 3$ is a half of a perfect code. But there are no any nontrivial binary perfect codes with $d \geq 3$, except mentioned above 1-perfect codes with Hamming parameters and the binary Golay $[23, 12, 7]$ code [19], [21]. We have, therefore, from the results above the following nonexistence result.

Theorem 3: Let C be a nontrivial (i.e. $|C| > 2$) non-antipodal completely regular binary code with parameters n, $d \geq 3$ and ρ. Then there are exactly two cases:
1). C is a half of binary perfect Golay code and $n = 23$, $d = 8$ and $\rho = 7$.
2). C is a half of binary perfect code with Hamming parameters, i. e. $n = 2^m - 1$, $d = 4$, $\rho = 3$, where $m \geq 3$.

Thus, from known binary perfect codes we obtain new completely regular codes taking halves of these codes. It comes from the following statement.

Theorem 4: Let C' be a binary nontrivial perfect code of (odd) length n, with minimal distance $2e'+1$ and such that $\mathbf{0} \in C'$. Denote by C the subcode of C', formed by all codewords of even (respectively, odd) weight. Then C is completely regular with covering radius $\rho = 2e'+1$ and with intersection array

$$(n, ..., n-e', e', ...1; 1, ..., e', n-e', ..., n).$$

Furthermore, if C' is completely transitive, then C is completely transitive too.

Proof. Let C be the even subcode of C', i.e. $\mathbf{0} \in C$. Since C' has minimal distance $2\rho' + 1$, clearly we have that $\rho(C) = \rho = 2e' + 1$. Write out the intersection numbers a_i, b_i, and c_i of C. Since $d = 2e' + 2$ and $\rho = 2e' + 1$, we have immediately for $i = 0, 1, \ldots, e'$:

$$a_i = 0, \;\; b_i = n - i, \;\; c_i = i.$$

But $C(\rho)$ is the shift of C by $\mathbf{1}$, so the numbers a_i, b_i, and c_i of C for $i = \rho, ..., e' + 1$ are inverse of those values for $i = 0, 1, \ldots, e'$ (Lemma 2), i.e.

$$a_i = 0, \;\; b_i = c_{\rho-i}, \;\; c_i = b_{\rho-i}.$$

Clearly these numbers do not depend on the choice of $\mathbf{x}$ and coincide with numbers given by the theorem. Thus, C is completely regular with intersection array given in the statement.

For the second statement consider the coset D' of C' of some weight i: say $D' = C' + \mathbf{x}$ where $\mathrm{wt}(\mathbf{x}) = i$ and $i \leq \rho'$. Clearly this coset consists of two subsets: $D' = D \cup D(\rho)$, where D is a coset of C of weight i and $D(\rho)$ is the shift of $C(\rho)$ by vector $\mathbf{x}$ of weight i. We note that, if D is even (respectively odd), then $D(\rho)$ is odd (respectively, even). This means that two sets D and $D(\rho)$ consist of codewords of different parities. We conclude, that these two sets are fixed under action of any permutation automorphism of C' (indeed, permutations do not change the weight of words). So, if D' runs over all different cosets of weight i, then under action of the same automorphisms, the coset D of C runs over the all different cosets of C of weight i. Assuming that C' is completely transitive, we obtain the same for C, taking into account that $\mathrm{Perm}(C)$ contains $\mathrm{Perm}(C')$ as a subgroup. $\triangle$

We have also a natural explanation based on antipodality of the well known result that any nontrivial binary perfect code has odd length n and odd covering radius ρ.

Theorem 5: Let C be a nontrivial binary e-error-correcting perfect code of length n. Then n and e are both odd.

Proof. Let $\mathbf{0} \in C$. As we know from [13], $\mathbf{1} \in C$. By Theorem 4 the even (or odd) half of C is a completely regular non-antipodal code. Then by Theorem 1 we deduce that $\rho = e$ is odd. Now let C^* be the even extension of C. It is well known [7], that C^* is completely regular with $\rho^* = \rho + 1$. By Lemma 5 the code C^* is antipodal, which is possible only if n is odd. △

We note that for trivial binary perfect codes with $|C| = 2$ the result is not valid. Indeed, in such code $n = 2e + 1$ is odd (how it should be), but e is any integer.

IV. New completely regular codes from binary perfect codes

Here we give new completely regular codes, which are obtained from known binary perfect codes, according to Theorem 4. These codes are also new completely transitive codes and new uniformly packed codes in the wide sense, i.e. in the sense of [1].

Theorem 6: Let G' be a binary perfect Golay $[23, 12, 7]$-code, which is completely regular with $\rho' = e' = 3$. Denote by G its subcode, formed by all codewords of even (respectively, odd) weight. Then:

(i) G is a completely transitive $[23, 11, 8]$-code with $\rho = 7$.

(ii) G is a completely regular code with intersection array $(23, 22, 21, 20, 3, 2, 1; 1, 2, 3, 20, 21, 22, 23)$.

(iii) G is a uniformly packed code in the wide sense with parameters: $\alpha_0 = 1$, $\alpha_1 = \frac{7}{23}$, $\alpha_2 = \frac{3}{11}$, $\alpha_3 = \frac{179}{7\cdot 11\cdot 23}$, $\alpha_4 = \frac{29}{5\cdot 7\cdot 11}$, $\alpha_5 = \frac{47}{7\cdot 11\cdot 23}$, $\alpha_6 = \frac{1}{7\cdot 11}$, $\alpha_7 = \frac{1}{11\cdot 23}$.

Proof. The intersection array comes from Theorem 4. We explain shortly how to find the parameters α_i of the new uniformly packed code. We use the defining equation (1). Recall that subsets G'_w of G' for $w = 7, 8, 11, 12, 15, 16$ are 4-designs $T_w(23, w, 4, \lambda(w))$ with values [14]

$$\lambda(7) = 1,\ \lambda(8) = 4,\ \lambda(11) = 48,\ \lambda(12) = 72. \qquad (2)$$

But any t-design with given λ is also j-design for any $j = 1, ..., t$ with λ_j which is easily computed [17]. For the values $\lambda(w)$ given in (2), denote by $\lambda_j(w)$ the corresponding values λ_j.

We start with α_0. Since $d(G) = 8$ we have that $\alpha_0 = 1$. For the case $\mathbf{v} \in G(1)$ the equation (1) becomes

$$\alpha_1 \cdot f_1(\mathbf{v}) + \alpha_7 \cdot f_7(\mathbf{v}) \ = \ 1,$$

where $f_j(\mathbf{v})$ denotes the number of words of G which are at the distance j from $\mathbf{v}$. Clearly $f_1(\mathbf{v}) = 1$ and $f_7(\mathbf{v}) = \lambda_1(8)$, implying that

$$\alpha_1 \ + \ 176 \cdot \alpha_7 \ = \ 1. \qquad (3)$$

For the case $\mathbf{v} \in G(7)$ the equation (1) becomes

$$\alpha_7 \cdot f_7(\mathbf{v}) \ = \ 1.$$

Since $G(\rho)_\rho = G'_7$, we have that $f_7(\mathbf{v}) = 253$, implying that $\alpha_7 = 1/253$, and hence, $\alpha_1 = 7/23$ by (3).

For the case $\mathbf{v} \in G(2)$ the equation (1) is

$$\alpha_2 \cdot 1 + \alpha_6 \cdot f_6(\mathbf{v}) \ = \ 1, \qquad (4)$$

where $f_2(\mathbf{v}) = 56$, since G_8 is a 2-design with $\lambda_2(8) = 56$.

Continuing in the same way, we obtain:

for the case $\mathbf{v} \in G(3)$:

$$\alpha_3 \cdot 1 + \alpha_5 \cdot \lambda_3(8) + \alpha_7 \cdot (3(\lambda_2(8) - \lambda_3(8))) \ = \ 1; \qquad (5)$$

for the case $\mathbf{v} \in G(4)$:

$$\alpha_4 \cdot (1 + \lambda_4(8)) + \alpha_6 \cdot (4(\lambda_3(8) - \lambda_4(8))) \ = \ 1; \qquad (6)$$

for the case $\mathbf{v} \in G(5)$:

$$\alpha_5 \cdot (1 + 5\lambda_4(8)) + \alpha_7 \cdot f_7(\mathbf{v}) \ = \ 1; \qquad (7)$$

for the case $\mathbf{v} \in G(6)$:

$$\alpha_6 \cdot f_6(\mathbf{v}) \ = \ 1. \qquad (8)$$

For the values $f_5(\mathbf{v})$ and $f_6(\mathbf{v})$ we have clearly

$$f_6(\mathbf{v}) \ = \ 1 + \binom{6}{2}\lambda_4(8) + \lambda_6(12) \ = \ 77. \qquad (9)$$

and

$$f_7(\mathbf{v}) \ = \ \frac{1}{2}\binom{5}{2}\lambda_3(8) \ + \ \lambda_5(12) \ = 112. \qquad (10)$$

In the last two expressions we used the intersection arrays of the corresponding designs (see [14]). From (8) we have that $\alpha_6 = 1/77$. Using this value for (4) - (7), we obtain the other values α_i, given in the statement. $\triangle$

We remark, that this new uniformly packed code G is the first example (known to the authors) of such code with eight different values of parameters α_i.

Theorem 7: Let H' be a binary perfect additive $[n,k,d]$-code, i.e. $n = 2^m - 1 \geq 7$, $k' = n - m$ and $d' = 3$, which is completely regular with $\rho' = e' = 1$. Denote by H its even (respectively, odd) subcode. Then:
(i) H is completely regular with covering radius $\rho = 3$ and with intersection array $(n, n-1, 1; 1, n-1, n)$.
(ii) H is uniformly packed in the sense of [1] with parameters α_i:

$$\alpha_0 = 1,\ \alpha_1 = \frac{3}{n},\ \alpha_2 = \frac{2}{n-1},\ \alpha_3 = \frac{6}{n(n-1)}.$$

Proof. Similarly to the previous theorem we show here only how to find the parameters α_i. Since $d = 4$ and $\rho = 3$ we have $\alpha_0 = 1$ and

$$\alpha_3 \ = \ \frac{1}{|H_3'|}. \qquad (11)$$

For α_1 we have immediately

$$\alpha_1 \ + \ \alpha_3 \cdot \frac{4}{n} \cdot |H_4| \ = \ 1,$$

implying that $\alpha_1 = 3/n$. Finally for α_2 we have

$$\alpha_2 \cdot \left(1 + \frac{n-3}{2}\right) \ = \ 1,$$

implying that $\alpha_2 = 2/(n-1)$. $\triangle$

As it is known from [18] the binary codes which are closest to binary perfect codes are Preparata-like codes. Unfortunately, a half of such code of length 15 is not even uniformly packed code in the wide sense.

From the results of papers [4] and [12] we know all the non-equivalent binary perfect $\mathbb{Z}_4$-linear and additive codes with $d = 3$. This implies the following result.

Theorem 8: For any $m \geq 4$ there exist exactly $\lfloor (m+1)/2 \rfloor$ non-equivalent extended $\mathbb{Z}_4$-linear and exactly $\lfloor m/2 \rfloor$ additive (non-$\mathbb{Z}_4$-linear) completely regular codes with $d = 4$, $\rho = 3$ and with intersection array

$$(n, n-1, 1; 1, n-1, n).$$

If the original codes are completely transitive, the new codes are completely transitive too.

Proof. Let C' and B' be two nonequivalent binary perfect codes of length n and $d = 3$. We have to prove that their halves, i.e. the codes C and B, are nonequivalent also. Assume in contrary that B and C are equivalent. This means that there is a vector $\mathbf{h} \in F^n$ and a permutation θ in the group of S_n (all permutations of n elements) such that:

$$B + \mathbf{h} \ = \ \theta(C). \qquad (12)$$

By Theorems 1 and 2 we have that $C' = C \cup C(\rho)$ and $B' = B \cup B(\rho)$. Now from (12) we obtain that

$$B + \mathbf{1} + \mathbf{h} \ = \ \theta(C) + \mathbf{1} \ = \ \theta(C + \mathbf{1}), \qquad (13)$$

where the last equality follows since $\mathbf{1}$ is fixed by any permutation from S_n. But $C(\rho) = C + \mathbf{1}$ as well as $B(\rho) = B + \mathbf{1}$ (Lemma 4). Adding (12) and (13) we obtain that the codes C' and B' are equivalent, i.e. a contradiction.

The second statement of theorem follows from Theorem 4. $\triangle$

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